Game Theory
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Lecture 6

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Outline

1. Stackelberg duopoly and the first mover's advantage
2. Formal definitions
3. Bargaining and discounted payoffs

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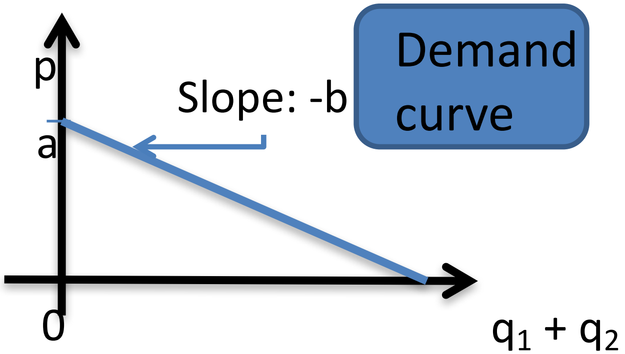
1. Stackelberg duopoly and the first mover's advantage
2. Formal definitions
3. Bargaining and discounted payoffs

Cournot Competition reminder

- The players: 2 Firms, e.g. Coke and Pepsi
- Strategies: quantities players produce of identical products: q_i, q_{-i}
  - Products are perfect substitutes
- The payoffs
  - Constant marginal cost of production c
  - Market clearing price: $p = a - b\,(q_1 + q_2)$
  - firms aim to maximize profit



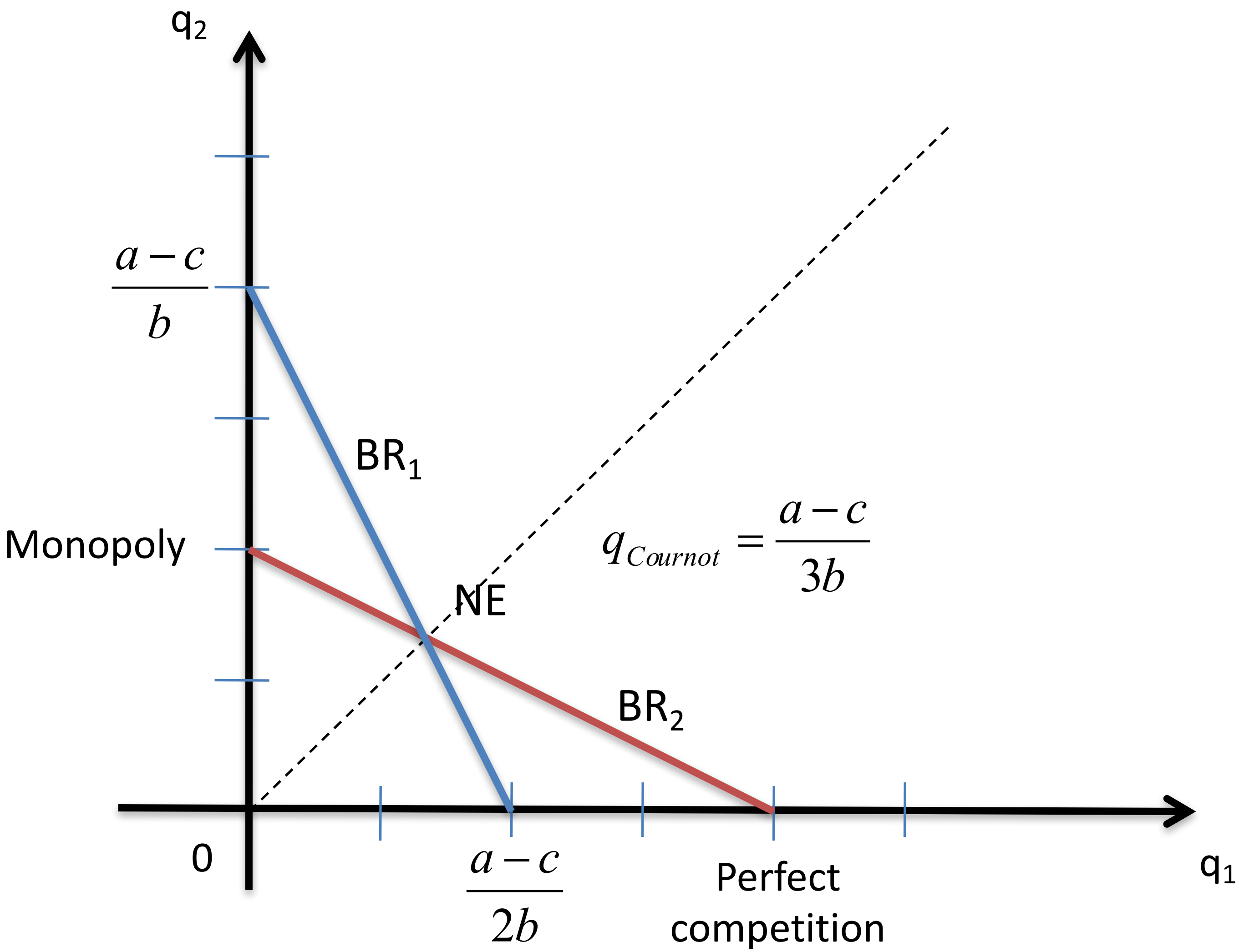
$$u_1(q_1,q_2) = p * q_1 - c * q_1$$

Nash equilibrium

- $u_1(q_1,q_2) = a * q_1 – b * q^2{}_1 – b * q_1\, q_2 – c * q_1$
- FOC, SOC give best responses:
$$\begin{cases} \hat{q}_1 = BR_1(q_2) = \dfrac{a-c}{2b} - \dfrac{q_2}{2} \\ \hat{q}_2 = BR_2(q_1) = \dfrac{a-c}{2b} - \dfrac{q_1}{2} \end{cases}$$
- NE is when they cross:
$$BR_1(q_2) = BR_2(q_1) \Rightarrow q_1^* = q_2^*$$
$$\frac{a-c}{2b} - \frac{q_2}{2} = \frac{a-c}{2b} - \frac{q_1}{2}$$
$$\Rightarrow q_1^* = q_2^* = \frac{a-c}{3b}$$

→ Cournot quantity

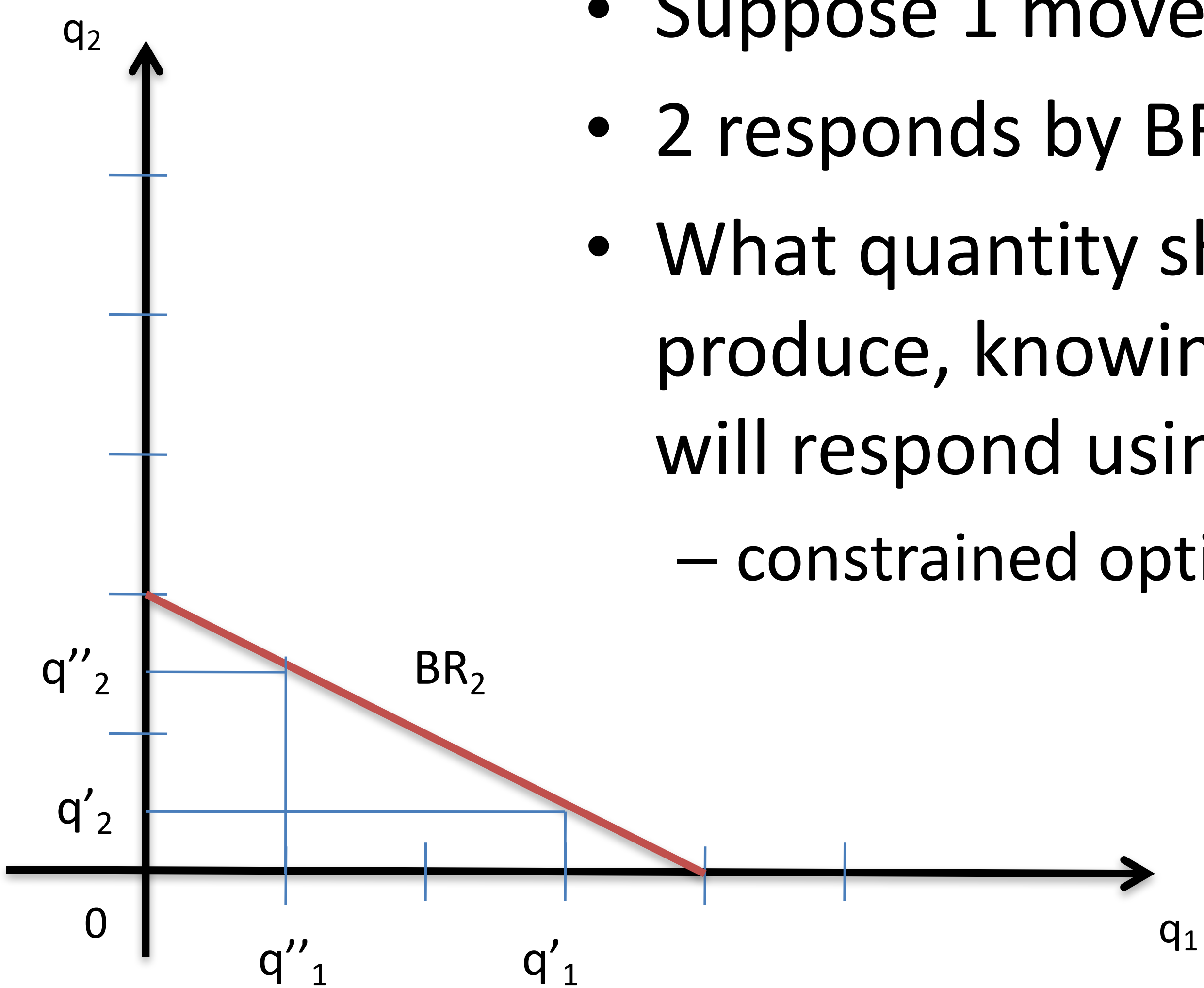
Graphically



Stackelberg Model

- Assume now that one firm gets to move first and the other moves after
  - That is one firm gets to set the quantity first
- Is it an advantage to move first?
  - Or it is better to wait and see what the other firm is doing and then react?
- We are going to use backward induction to compute the quantities
  - We cannot draw trees here because of the continuum of possible actions

Intuition



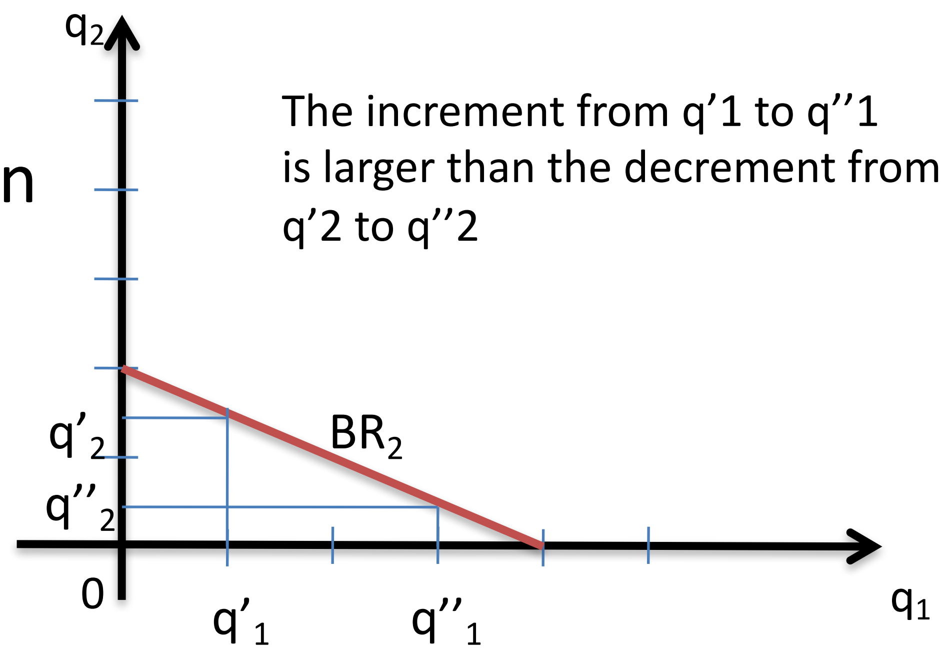
- Suppose 1 moves first
- 2 responds by BR ! (by def)
- What quantity should firm 1 produce, knowing that firm 2 will respond using the BR?
  - constrained optimization problem

Intuition (2)

- Should firm 1 produce more or less than the Cournot quantity?
  - Products are strategic substitutes: the more firm 1 produces, the less firm 2 will produce and vice-versa
  - Firm 1 producing more ➔ firm 1 is happy
- What happens to firm 1's profits?
  - They go up, otherwise firm 1 wouldn't have set higher production quantities
- What happens to firm 2's profits?
  - The answer is not immediate
- What happened to the total output in the market?
  - Even here the answer is not immediate

Intuition (3)

- What happened to the total output in the market?
  - Consumers would like the total output to go up, for that would mean that prices would go down!
  - Indeed, it goes down: see the BR curve



Intuition (4)

- What happens to firm 2's profits?
  - q1 went up, q2 went down
  - q1+q2 went up ➔ prices went down
  - Firm 2's costs are the same

➔Firm 2's profit went down

- We have seen that firm 1's profit goes up

➔Conclusion: First mover is an asset (here!)

Stackelberg Model computations

- Let us now compute the quantities. We have

$$p = a - b(q_1 + q_2)$$

$$\text{profit}_i = pq_i - cq_i$$

- We apply the Backward Induction principle
  - First, solve the maximization problem for firm 2, taking q1 as given
  - Then, focus on firm 1

Stackelberg Model computations (2)

- Firm 2's optimization problem (for fixed q1)

$$\max_{q_2}\left[\left(a-bq_1-bq_2\right)q_2-cq_2\right]$$

$$\frac{\partial}{\partial q_2}\Rightarrow q_2=\frac{a-c}{2b}-\frac{q_1}{2}$$

- We now can take this quantity and plug it in the maximization problem for firm 1

Stackelberg Model computations (3)

- Firm 1's optimization problem:

$$\max_{q_1}\left[\left(a - bq_1 - bq_2\right)q_1 - cq_1\right] =$$

$$\max_{q_1}\left[\left(a - bq_1 - b\left(\frac{a-c}{2b} - \frac{q_1}{2}\right)\right) - c\right]q_1 =$$

$$\max_{q_1}\left[\frac{a-c}{2} - \frac{bq_1}{2}\right]q_1 = \max_{q_1}\left[\frac{a-c}{2}q_1 - b\frac{q_1^2}{2}\right]$$

Stackelberg Model computations (4)

- We derive F.O.C. and S.O.C.

$$\frac{\partial}{\partial q_1} = 0 \Rightarrow \frac{a-c}{2} - bq_1 = 0$$

$$\frac{\partial^2}{\partial q_1^2} = -b < 0$$

- This gives us

$$q_1 = \frac{a-c}{2b}$$

$$q_2 = \frac{a-c}{2b} - \frac{1}{2}\frac{a-c}{2b} = \frac{a-c}{4b}$$

Stackelberg quantities

- All this math to verify our initial intuition!

$$q_1^{NEW} > q_1^{Cournot}$$

$$q_2^{NEW} < q_2^{Cournot}$$

$$q_1^{NEW} + q_2^{NEW} = \frac{3(a-c)}{4b} > \frac{2(a-c)}{3b} = cournot$$

Observations

- Is what we've looked at really a sequential game?
  - Despite we said firm 1 was going to move first, there's no reason to assume she's really going to do so!

- We need a ***<u>commitment</u>***
- In this example, sunk cost could help in believing firm 1 will actually play first

➔ Assume for instance firm 1 was going to invest a lot of money in building a plant to support a large production: this would be a credible commitment!

Simultaneous vs. Sequential

- There are some key ideas involved here

1. Games being simultaneous or sequential is ***not really about timing, it is about information***
2. Sometimes, more information can hurt!
3. Sometimes, more options can hurt!

First mover advantage

- Advocated by many "economics books"
- Is being the first mover always good?
  - ***Yes, sometimes***: as in the Stackelberg model
  - ***Not always***, as in the Rock, Paper, Scissors game
  - Sometimes neither being the first nor the second is good, as in the "I split you choose" game

The NIM game

- We have two players
- There are two piles of stones, A and B
- Each player, in turn, decides to delete some stones from whatever pile
- The player that remains with the last stone wins

The NIM game (2)

- If piles are equal ➔ second mover advantage
  - You want to be player 2
- If piles are unequal ➔ first mover advantage
  - You want to be player 1
  - Correct tactic: You want to make piles equal

- You know who will win the game from the initial setup
- You can solve through backward induction

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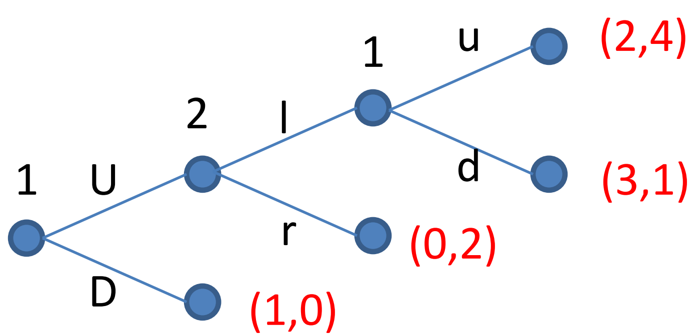
1. Stackelberg duopoly and the first mover's advantage
2. Formal definitions
3. Bargaining and discounted payoffs

Perfect Information and pure strategy

A game of ***perfect information*** is one in which at each node of the game tree, the player whose turn is to move knows which node she is at and how she got there

A ***pure strategy*** for player *i* in a game of perfect information is a ***complete plan*** of actions: it specifies which action *i* will take at each of its decision nodes

Example

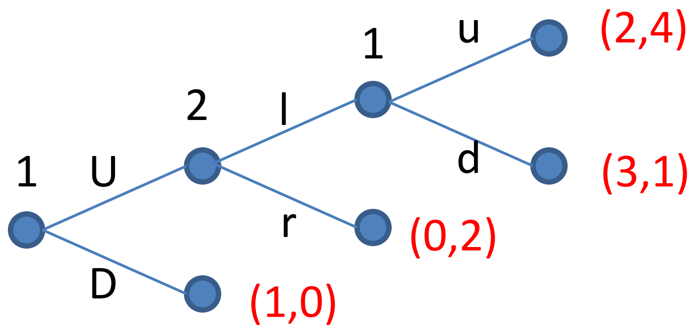


- Strategies
  - Player 2: [l], [r]
  - Player 1: [U,u], [U,d] [D, u], [D,d]

look redundant!

- Note:
  - In this game it appears that player 2 may never have the possibility to play her strategies
  - This is also true for player 1!

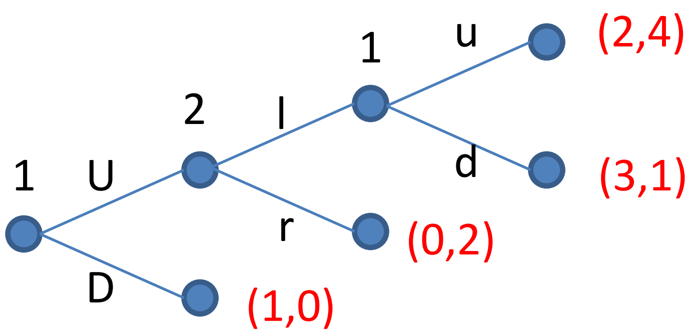
Backward induction solution



- Backward Induction
  - Start from the end
    - “d” → higher payoff
  - Summarize game
    - “r” → higher payoff
  - Summarize game
    - “D” → higher payoff

- BI :: {[D,d],r}

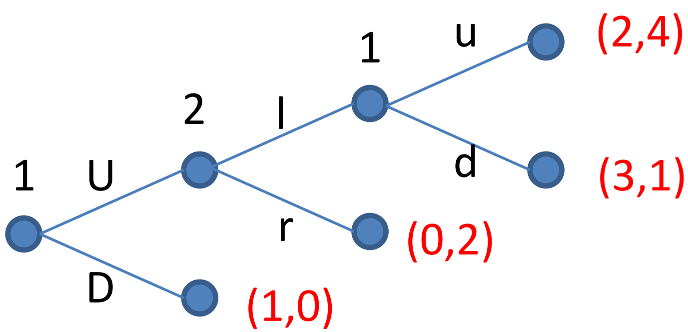
Transformation to normal form



	l	r
U u	**2,4**	**0,2**
U d	3,1	0,2
D u	1,0	1,0
D d	1,0	1,0

From the ***extensive form***
To the ***normal form***

Backward induction versus NE



	l	r
U u	**2,4**	**0,2**
U d	3,1	0,2
D u	1,0	1,0
D d	1,0	1,0

Backward Induction

{[D, d],r}

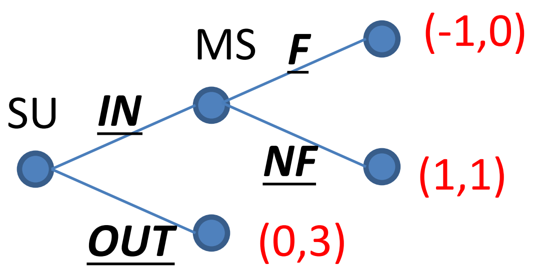
Nash Equilibrium

{[D, d],r}
{[D, u],r}

A Market Game (1)

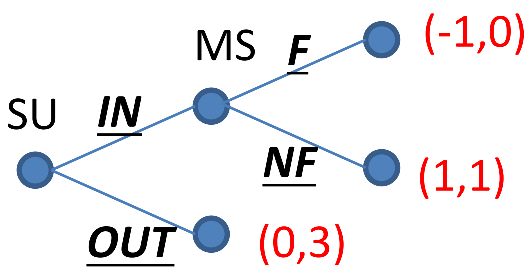
- Assume there are two players
  - An incumbent monopolist (MicroSoft, MS) of O.S.
  - A young start-up company (SU) with a new O.S.
- The strategies available to SU are:
  Enter the market (***IN***) or stay out (***OUT***)
- The strategies available to MS are:
  Lower prices and do marketing (***FIGHT***) or stay put (***NOT FIGHT***)

A Market Game (2)



- What should you do?
- Analyze the game with BI
- Analyze the normal form equivalent and find NE

A Market Game (3)



	F	NF
IN	**-1,0**	**1,1**
OUT	0,3	0,3

Backward Induction

(IN, NF)

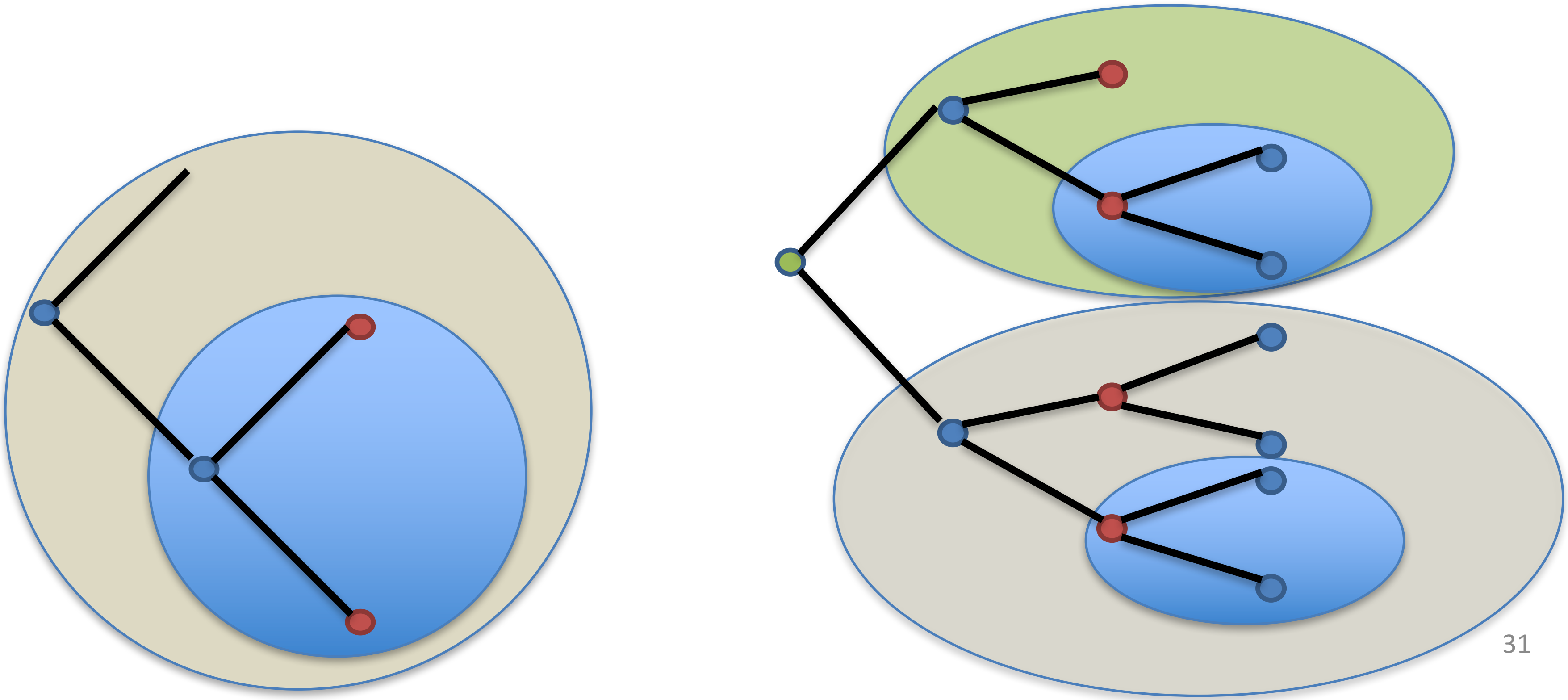
Nash Equilibrium

(IN, NF)
(OUT, F)

- (OUT, FIGHT) is a NE but relies on an incredible threat
  - Introduce subgame perfect equilibrium

Sub-games

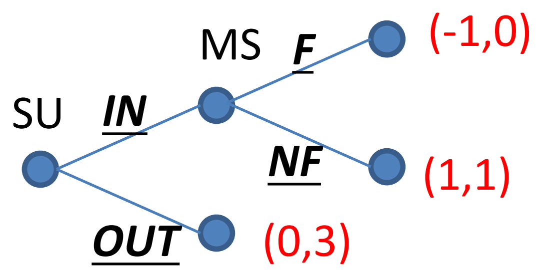
- A **sub-game** is a part of the game that looks like a game within the tree. It starts from a single node and comprises all successors of that node



sub-game perfect equilibrium (SPE)

- A Nash Equilibrium (s_1*,s_2*,…,s_N*) is a sub-game perfect equilibrium if it induces a Nash Equilibrium in every sub-game of the game

- Example:
  - (IN, NF) is a SPE
  - (OUT, F) is not a SPE
    - Incredible threat



	F	NF
IN	**-1,0**	**1,1**
OUT	0,3	0,3

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Ultimatum game

- Two players, player 1 is going to make a "take it or leave it" offer to player 2
- Player 1 is given a pie worth $1 and has to decide how to divide it
  - *(S, 1-S)*, e.*g. ($0.75, $0.25)*
- Player 2 has two choices: accept or decline the offer
- Payoffs:
  - If player 2 accepts: Player 1 gets *S*, player 2 gets *1-S*
  - If player 2 declines: Player 1 and player 2 get nothing

- It doesn't look like real bargaining, but… let's play

Analysis with backward induction

- Start with the receiver of the offer, choosing to accept or refuse *(1-S)*
  – Assuming player 2 is trying to maximize her profit, what should she do?

- So, what should player 1 offer?

Prediction vs reality

- Is there a good match between backward induction prediction and what we observe?
- Why?

- Reasons why player 2 may reject:
  - Pride
  - She may be sensitive to how her payoffs relates to others
  - Indignation
  - Player 2 may want to “teach” a lesson to Player 1 to offer more
    - What we really played is a one-shot game but if we have played more than once, by rejecting an offer, player 2 would also induce player 1 to obtain nothing, which may be an incentive for player 1 to offer more in the next round of the game

- Why is the 50-50 split focal here?

Two-period bargaining game

- Two players, player 1 is going to make a "take it or leave it" offer to player 2
- Player 1 is given a pie worth $1 and has to decide how to divide it: (S_1, 1-S_1)
- Player 2 has two choices: accept or decline the offer
  - If player 2 accepts: Player 1 gets S_1, player 2 gets $1\text{-}S_1$
  - If player 2 declines: we flip the roles and play again
    - This is the second stage of the game
- The second stage is exactly the ultimatum game: player 2 chooses a division (S_2, 1-S_2)
- Player 1 can accept or reject
  - If player 1 accepts, the deal is done
  - If player 1 rejects, none of them gets anything

Discount factor

- Now, we add one important element
  - In the first round, the pie is worth \$1
  - If we end up in the second round, **the pie is worth less**

- Example:
  - If I give you \$1 today, that's what you get
  - If I give you \$1 in 1 month, we assume it's worth less, say

    $\delta < 1$

- **Discounting factor**:
  - From today perspective, \$1 tomorrow is worth $\delta < 1$

Game analysis idea

- It is clear that the decision to accept or reject partly depends on what you think the other side is going to do in the second round

➔ This is **backward induction**!

  – By working backwards, we can see that what you should offer in the first round should be just enough to make sure it's accepted, knowing that the person who's receiving the offer in the first round is going to think about the offer they're going to make you in the second round, and they're going to think about whether you're going to accept or reject

Two-period bargaining game analysis

- Let’s analyze the game formally with **backward induction**
  - We ignore any “pride” effect

- One stage game (the ultimatum game)

	Offerer’s split	Receiver’s split
1-period	*1*	*0*

Two-period bargaining game analysis (2)

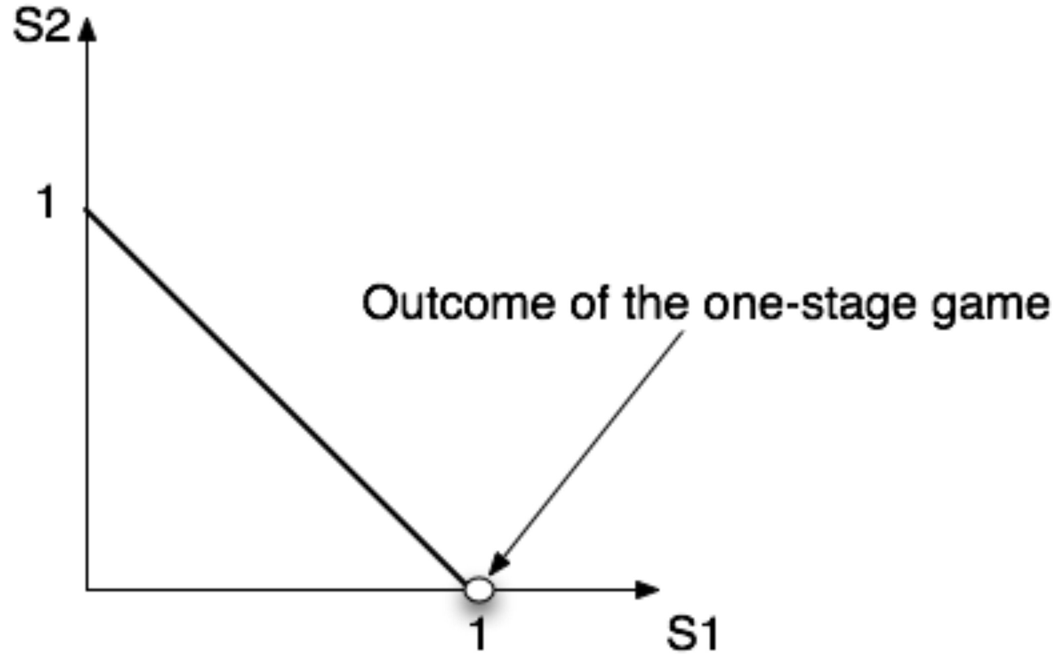
- Two-stage game

	Offerer's split	Receiver's split
1-period	*1*	*0*
2-period	$1-\delta$	$\delta < 1$

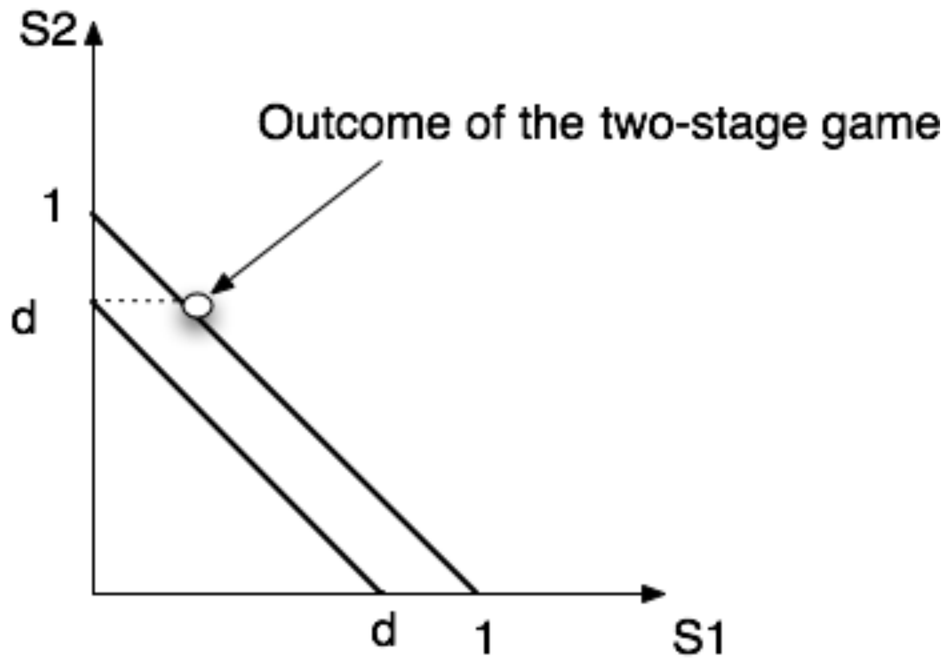
Let's be careful:

- In the second round of the two-period game, player 2 makes the offer about the whole pie
- We know that this is going to be an ultimatum game, so player 2 will keep the whole pie and player 1 will accept (by BI)
- However, seen from the first round, the pie in the second round that player 2 could get, is worth less than $1

Two-period bargaining game graphically



Two-period bargaining game graphically (2)



Three-period bargaining game

- The rules are the same as for the previous games, but now there are two possible flips
  - Period 1: player 1 offers first
  - Period 2: if player 2 rejected the offer in period 1, she gets to offer
  - Period 3: if player 1 rejected the offer in period 2, he gets to offer again
- ***NOTE***: the value of the pie keeps shrinking
  - It's not the pie that really shrinks, it's that we assumed players are **discounting**

Three-period bargaining game analysis

- **Discounting**: the value to player 1 of a pie in round three is discounted by $\delta \cdot \delta = \delta^2$
- Analysis with **backward induction**
  – Again, assume "no pride"
  – We start from round three, which is our ultimatum game and we know there that player 1 can get the whole pie, since player 2 will accept the offer

➔ Player 1 could get a pie worth δ^2

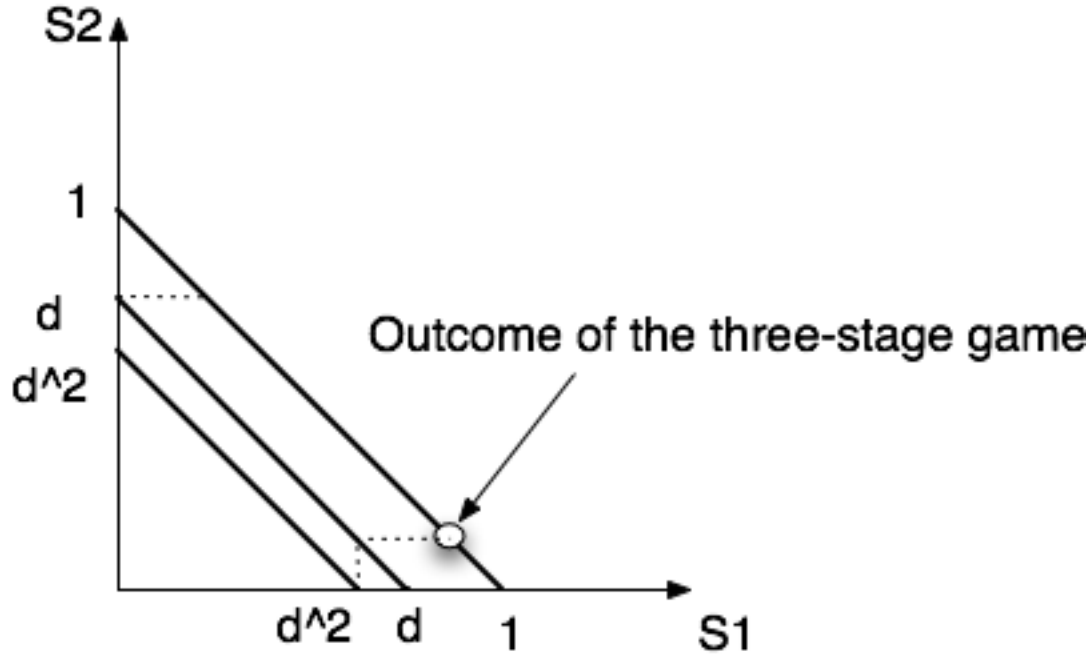
Three-period bargaining game result

- Three-period game

	Offerer's split	Receiver's split
1-period	*1*	*0*
2-period	$1-\delta$	$\delta < 1$
3-period	$1-\delta(1-\delta)$	$\delta(1-\delta)$

- **NOTE**: in the table, we report the split player 1 should offer in the first round of the game
- In the first round, if the offer is rejected, we go into a 2-period game, and we know what the split is going to look like

Three-period bargaining game graphically



Four-periods

- What about a 4-period bargaining game?

	Offerer	Receiver
1-period	*1*	*0*
2-period	$1-\delta$	$\delta < 1$
3-period	$1-\delta(1-\delta)$	$\delta(1-\delta)$
4-period	*?*	*?*

- **NOTE**: give people just enough today so they'll accept the offer, and just enough today is whatever they get tomorrow discounted by delta
- You don't need to go back all the way up to period 1

Four-periods result

- Let's clear out the algebra

	Offerer	Receiver
1-period	*1*	*0*
2-period	$1-\delta$	δ
3-period	$1-\delta+\delta^2$	$\delta-\delta^2$
4-period	$1-\delta+\delta^2-\delta^3$	$\delta-\delta^2+\delta^3$

n-periods

- Geometric series with reason (-δ)
- For example, player 1's share for n=10:

$$S_1^{(10)} = 1 - \delta + \delta^2 - \delta^3 + \delta^4 + \ldots - \delta^9 = \frac{1 - \left(-\delta\right)^{10}}{1 - (-\delta)} = \frac{1 - \delta^{10}}{1 + \delta}$$

Some observations

- In the one-stage game, there's a huge first-mover advantage
- In the two-stage game, its more difficult: it depends on how large is delta. If it is large, you'd prefer being the receiver
- In the three-stage game it looks like you'd be better off by making the offer, but again it's not very easy
- What about the 10-stage game? It seems that the two players are getting closer in terms of payoffs, and that the initial bargaining power has diminished

Large number of periods

- Let's look at the asymptotic behavior of this game, when there is an infinite number of stages

$$S_1^{(\infty)} = \frac{1-\delta^{\infty}}{1+\delta} = \frac{1}{1+\delta}$$

$$S_2^{(\infty)} = 1 - S_1^{(\infty)} = \frac{\delta+\delta^{\infty}}{1+\delta} = \frac{\delta}{1+\delta}$$

Discount factor close to one

- Now, let's imagine that the offers are made in **rapid succession**: this would imply that the discount factor we hinted at is almost negligible, which boils down to assume delta to be very close to 1

$$S_1^{(\infty)} = \frac{1}{1+\delta} \xrightarrow{\delta \approx 1} \frac{1}{2}$$

$$S_2^{(\infty)} = \frac{\delta}{1+\delta} \xrightarrow{\delta \approx 1} \frac{1}{2}$$

- So, if we assume rapidly alternating offers, we end up with a 50-50 split!