Game Theory
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Lecture 4

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Lecture 2-3 recap

- Proved existence of pure strategy Nash equilibrium in games with compact convex action sets and continuous concave utilities
- Defined mixed strategy Nash equilibrium
- Proved existence of mixed strategy Nash equilibrium in finite games
- Discussed computation and interpretation of mixed strategies Nash equilibrium

→ Nash equilibrium is not the only solution concept

→ Today: Another solution concept: evolutionary stable strategies

Outline

- Evolutionary stable strategies

Evolutionary game theory

- Game theory ←→ evolutionary biology
- Idea:
  - *Relate strategies to phenotypes* of genes
  - *Relate payoffs to genetic fitness*
  - Strategies that do well "grow", those that obtain lower payoffs "die out"
- Important note:
  - Strategies are ***hardwired***, they are not chosen by players
- Assumptions:
  - Within species competition: no mixture of population

Examples

- Using game theory to understand population dynamics
  - Evolution of species
  - Groups of lions deciding whether to attack in group an antelope
  - Ants deciding to respond to an attack of a spider

  - TCP variants, P2P applications

- Using evolution to interpret economic actions
  - Firms in a competitive market
  - Firms are bounded, they can’t compute the best response, but have rules of thumbs and adopt hardwired (consistent) strategies

  - Survival of the fittest == rise of firms with low costs and high profits

A simple model

- Assume simple game: two-player symmetric
- Assume **random tournaments**
  - Large population of individuals with hardwired strategies, pick two individuals at random and make them play the symmetric game
  - The player adopting the strategy yielding higher payoff will survive (and eventually gain new elements) whereas the player who "lost" the game will "die out"

- Start with entire population playing strategy *s*
- Then introduce a ***mutation***: a ***small*** group of individuals start playing strategy *s'*
- Question: will the mutants survive and grow or die out?

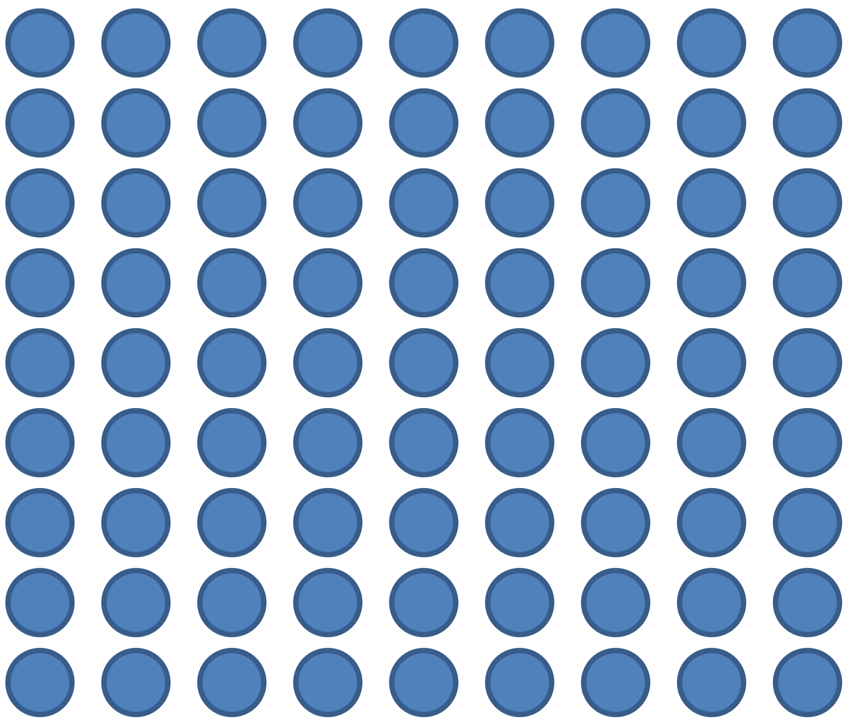
A simple example (1)

		Player 2 Cooperate	Defect
Player 1	C	**2,2**	**0,3**
	D	3,0	1,1
		ε	$1-\varepsilon$

- Have you already seen this game?
- Examples:
  - Lions hunting in a cooperative group
  - Ants defending the nest in a cooperative group
- Question: ***is cooperation evolutionary stable***?

A simple example (2)

Player strategy hardwired → C

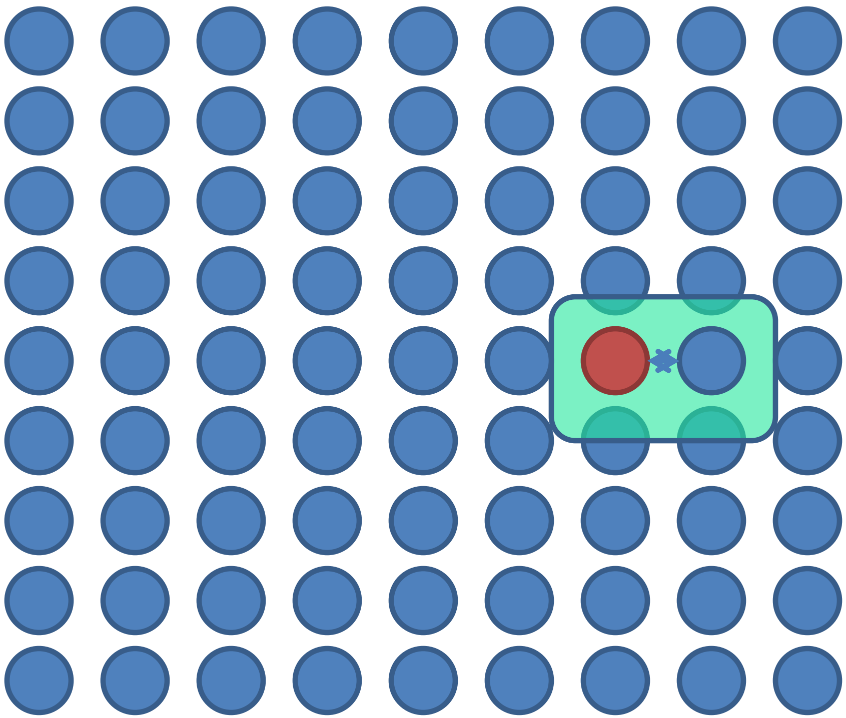


<u>"Spatial Game"</u>

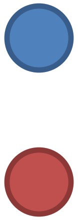
All players are cooperative and get a payoff of 2

What happens with a mutation?

A simple example (3)



Player strategy hardwired ➔ C



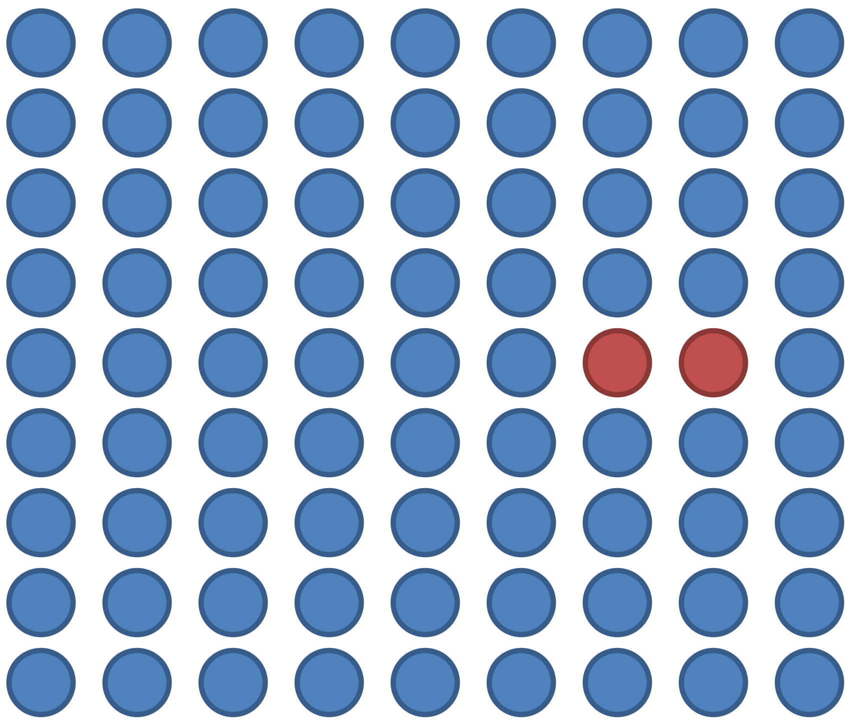
Player strategy hardwired ➔ D

Focus your attention on this random "tournament":

- Cooperating player will obtain a payoff of 0
- Defecting player will obtain a payoff of 3

Survival of the fittest:
D wins over C

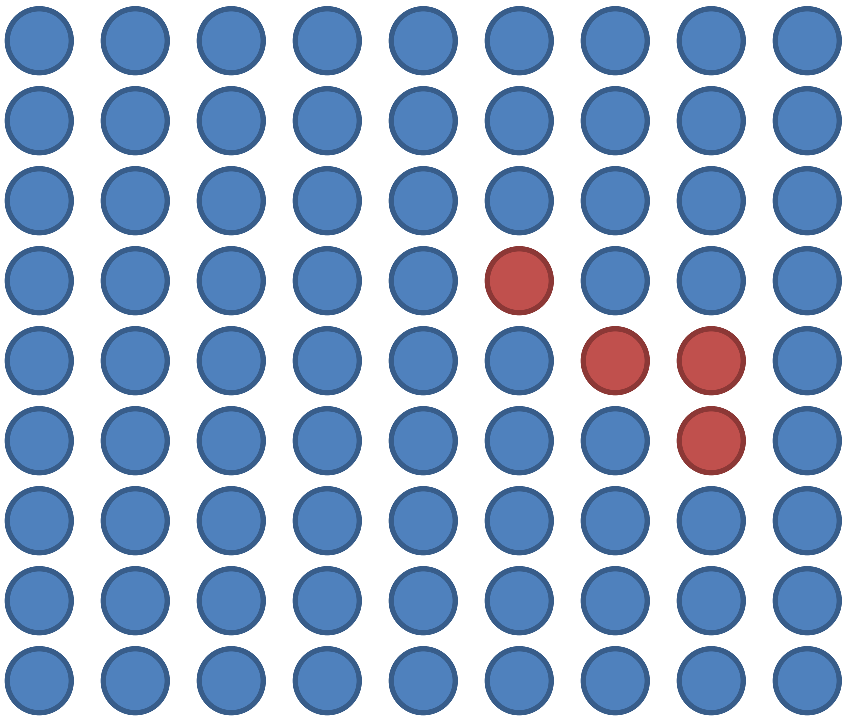
A simple example (4)



Player strategy hardwired ➔ C

Player strategy hardwired ➔ D

A simple example (5)



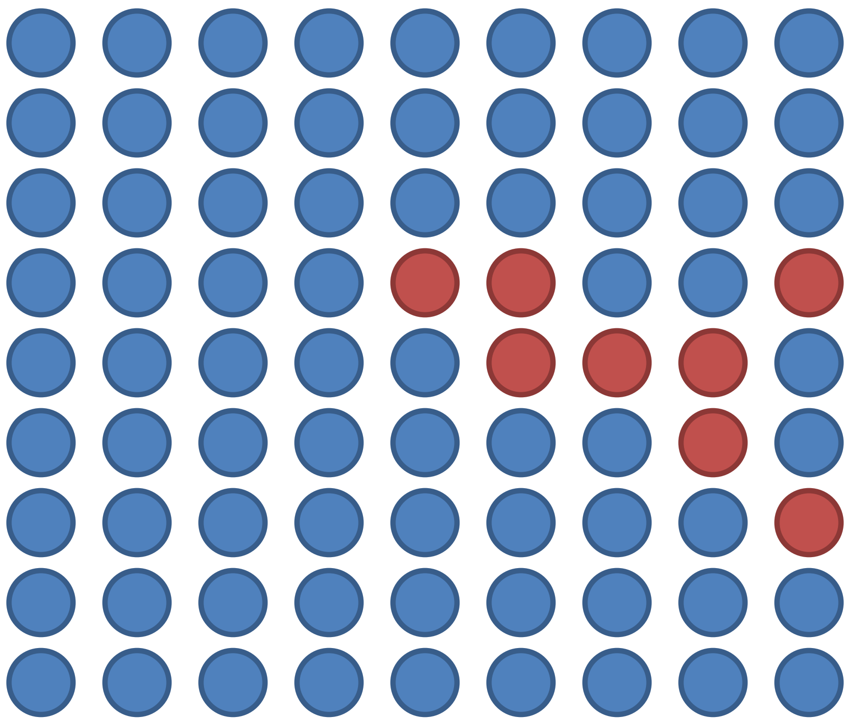
Player strategy hardwired → C

Player strategy hardwired → D

A simple example (6)

Player strategy hardwired → C

Player strategy hardwired → D



A small initial mutation is rapidly expanding instead of dying out

Eventually, C will die out

→ Conclusion: **C is not ES**

Remark: we have assumed asexual reproduction and no gene redistribution

ESS Definition 1 [Maynard Smith 1972]

Definition 1: Evolutionary stable strategy

In a symmetric 2-player game, the pure strategy ŝ is ES (in pure strategies) if there exists $\varepsilon_0 > 0$ such that:

$$\underbrace{(1-\varepsilon)\left[u(\hat{s},\hat{s})\right]+\varepsilon\left[u(\hat{s},s')\right]}_{\text{Payoff to ES } \hat{s}} > \underbrace{(1-\varepsilon)\left[u(s',\hat{s})\right]+\varepsilon\left[u(s',s')\right]}_{\text{Payoff to mutant } s'}$$

for all possible deviations s’ and for all mutation sizes $\varepsilon < \varepsilon_0$.

ES strategies in the simple example

		Player 2 Cooperate	Player 2 Defect
Player 1	C	**2,2**	**0,3**
	D	3,0	1,1

	1- ε	ε	For C being a majority
	ε	1- ε	For D being a majority

- Is cooperation ES?

C vs. [(1-ε)C + εD] → (1-ε)2 + ε0 = 2(1-ε)

D vs. [(1-ε)C + εD] → (1-ε)3 + ε1 = 3(1-ε)+ ε

3(1-ε)+ ε > 2(1-ε)

→**C is not ES** because the average payoff to C is lower than the average payoff to D

→A **strictly dominated** is **never Evolutionarily Stable**

– The strictly dominant strategy will be a successful mutation

ES strategies in the simple example

		Player 2 Cooperate	Player 2 Defect	
Player 1	C	**2,2**	**0,3**	
	D	3,0	1,1	
		1- ε	ε	For C being a majority
		ε	1- ε	For D being a majority

- Is defection ES?

D vs. [εC + (1-ε)D] → (1-ε)1 + ε3 = (1-ε)+3ε

C vs. [εC + (1-ε)D] → (1-ε)0 + ε2 = 2ε

(1-ε)+3 > 2 ε

→ **D is ES**: any mutation from D gets wiped out!

Another example (1)

	a	b	c
a	**2,2**	**0,0**	**0,0**
b	0,0	0,0	1,1
c	0,0	1,1	0,0

- 2-players symmetric game with 3 strategies
- Is “c” ES? c vs. $[(1-\varepsilon)c + \varepsilon b] \rightarrow (1-\varepsilon)\ 0 + \varepsilon\ 1 = \varepsilon$
  b vs. $[(1-\varepsilon)c + \varepsilon b] \rightarrow (1-\varepsilon)\ 1 + \varepsilon\ 0 = 1-\varepsilon > \varepsilon$

➔ “c” is not evolutionary stable, as “b” can invade it

- Note: “b”, the invader, is itself not ES!
  - **It is not necessarily true that an invading strategy must itself be ES**
  - But it still avoids dying out completely (grows to 50% here)

Another example (3)

	a	b	c
a	**2,2**	**0,0**	**0,0**
b	0,0	0,0	1,1
c	0,0	1,1	0,0

- Is (c,c) a NE?

Observation

- If *s* is **not Nash** (that is *(s,s)* is not a NE), then *s* is **not evolutionary stable** (ES)

Equivalently:

- If *s* **is ES**, then *(s,s)* **is a NE**

- Question: is the opposite true? That is:
  - If *(s,s)* **is a NE**, then *s* **is ES**

Yet another example (1)

		Player 2	
		a	b
Player 1	a	**1,1**	**0,0**
	b	0,0	0,0
		ε	1- ε

- NE of this game: *(a,a)* and *(b,b)*
- Is *b* ES? b → 0

  a → (1-ε) 0 + ε 1 = ε > 0

➔ *(b,b)* is a NE, but it is not ES!

- This relates to the idea of a weak NE

➔ If *(s,s)* is a ***strict NE*** then *s* is ES

Strict Nash equilibrium

Definition: Strict Nash equilibrium

A strategy profile $(s_1^*, s_2^*, \ldots, s_N^*)$ is a strict Nash Equilibrium if, for each player i,

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*) \text{ for all } s_i \neq s_i^*$$

- Weak NE: the inequality is an equality for at least one alternative strategy
- Strict NE is sufficient but not necessary for ES

ESS Definition 2

Definition 2: Evolutionary stable strategy

In a symmetric 2-player game, the pure strategy ŝ is ES (in pure strategies) if:

A)

$$(\hat{s},\hat{s}) \text{ is a symmetric Nash Equilibrium}$$

$$u(\hat{s},\hat{s}) \geq u(s',\hat{s}) \quad \forall s'$$

AND

B)

$$\text{if } u(\hat{s},\hat{s}) = u(s',\hat{s}) \text{ then}$$

$$u(\hat{s},s') > u(s',s')$$

Link between definitions 1 and 2

Theorem

Definition 1 $\Leftrightarrow$ Definition 2

- Proof sketch:

Recap: checking for ES strategies

- We have seen a definition that connects Evolutionary Stability to Nash Equilibrium
- By def 2, to check that $\hat{s}$ is ES, we need to do:

  - First check if $(\hat{s},\hat{s})$ is a ***symmetric*** Nash Equilibrium
  - If it is a ***strict*** NE, we're done
  - Otherwise, we need to compare how $\hat{s}$ performs against a mutation, and how a mutation performs against a mutation
  - If $\hat{s}$ performs better, then we're done

Example: Is "a" evolutionary stable?

		Player 2	
		a	b
Player 1	a	**1,1**	**1,1**
	b	1,1	0,0
		ε	1- ε

- Is (a, a) a NE? Is it strict?
- Is "a" evolutionary stable?

Evolution of social convention

- Evolution is often applied to social sciences
- Let's have a look at how driving to the left or right hand side of the road might evolve

	L	R
L	**2,2**	**0,0**
R	0,0	1,1

- What are the NE? are they strict? What are the ESS?

- Conclusion: we can have several ESS
  - They need not be equally good

The game of Chicken

	a	b
a	**0,0**	**2,1**
b	1,2	0,0

- This is a **symmetric coordination game**
- Biology interpretation:
  - "a" : individuals that are aggressive
  - "b" : individuals that are non-aggressive
- What are the pure strategy NE?
  - They are not symmetric → no candidate for ESS

The game of Chicken: mixed strategy NE

	a	b
a	**0,0**	**2,1**
b	1,2	0,0

- What's the mixed strategy NE of this game?
  - Mixed strategy NE = [(2/3, 1/3) , (2/3 , 1/3)]

  ➔ This is a ***symmetric*** Nash Equilibrium

➔Interpretation: there is an equilibrium in which 2/3 of the genes are aggressive and 1/3 are non-aggressive

- Is it a strict Nash equilibrium?
- Is it an ESS?

Remark

- A mixed-strategy Nash equilibrium (with a support of at least 2 actions for one of the players) can never be a strict Nash equilibrium
- The definition of ESS is the same!

ESS Definition 2bis

Definition 2: **Evolutionary stable strategy**

In a symmetric 2-player game, the mixed strategy ŝ is ES (in mixed strategies) if:

A) $(\hat{s},\hat{s})$ is a symmetric Nash Equilibrium

$$u(\hat{s},\hat{s}) \geq u(s',\hat{s}) \quad \forall s'$$

AND

B) if $u(\hat{s},\hat{s}) = u(s',\hat{s})$ then

$$u(\hat{s},s') > u(s',s')$$

The game of Chicken: ESS

	a	b
a	**0,0**	**2,1**
b	1,2	0,0

- Mixed strategy NE = [(2/3, 1/3) , (2/3 , 1/3)].
- Is it an ESS? we need to check for all possible mixed mutations s': $u(\hat{s}, s') > u(s', s') \quad \forall s' \neq \hat{s}$
- Yes, it is (do it at home!)

- In many cases that arise in nature, the only equilibrium is a mixed equilibrium
  - It could mean that the gene itself is randomizing, which is plausible
  - It could be that there are actually two types surviving in the population (cf. our interpretation of mixed strategies)

Hawks and doves

Dove

Hawk

The Hawks and Dove game (1)

	H	D
H	**(v-c)/2, (v-c)/2**	**v,0**
D	0, v	v/2, v/2

- More general game of aggression vs. non-aggression
  - The prize is food, and its value is $v > 0$
  - There's a cost for fighting, which is $c > 0$
- Note: we're still in the context of ***within spices competition***
  - So it's not a battle against two different animals, hawks and doves, we talk about strategies
    - "Act dovish vs. act hawkish"
- What are the ESS? How do they change with c, v?

The Hawks and Dove game (2)

	H	D
H	**(v-c)/2, (v-c)/2**	**v,0**
D	0, v	v/2, v/2

- Can we have a ES population of doves?
- Is *(D,D)* a NE?
  - No, hence "D" is not ESS
  - Indeed, a mutation of hawks against doves would be profitable in that it would obtain a payoff of v

The Hawks and Dove game (3)

	H	D
H	**(v-c)/2, (v-c)/2**	**v,0**
D	0, v	v/2, v/2

- Can we have a ES population of Hawks?
- Is *(H,H)* a NE? It depends: it is a symmetric NE if $(v-c)/2 \geq 0$

- Case 1: v>c ➔ *(H,H)* is a ***strict*** NE ➔ *"H"* is ESS
- Case 2: v=c ➔ (v-c)/2 = 0 ➔ *u(H,H) = u(D,H)* -- (H, H) is a weak NE
  - Is *u(H,D) = v* larger than *u(D,D) = v/2? Yes* ➔ *"H"* is ESS

➔ H is ESS if $v \geq c$

- If the prize is high and the cost for fighting is low, then you'll see fights arising in nature

The Hawks and Dove game (4)

	H	D
H	**(v-c)/2, (v-c)/2**	**v,0**
D	0, v	v/2, v/2
	$\hat{s}$	$1-\hat{s}$

- What if $c > v$?
  - "H" is not ESS and "D" is not ESS (they are not NE)
- Step 1: find a mixed NE
- Step 2: verify the ESS condition

The Hawks and Dove game: results

- In case v < c we have an evolutionarily stable state in which we have v/c hawks
  1. As v ↗ we will have more hawks in ESS
  2. As c ↗ we will have more doves in ESS

- By measuring the proportion of H and D, we can get the value of v/c

- Payoff: $E\left[u(D,\hat{s})\right] = E\left[u(H,\hat{s})\right] = 0\frac{v}{c} + \left(1 - \frac{v}{c}\right)\frac{v}{2}$

One last example (1)

	R	P	S
R	**1,1**	**v,0**	**0,v**
P	0,v	1,1	v,0
S	v,0	0,v	1,1

- Assume 1<v<2
  - ~ Rock, paper, scissors
- Only NE: ŝ = (1/3,1/3,1/3) – mixed, not strict
- Is it an ESS?
  - Suppose s'=R
  - u(ŝ, R) = (1+v)/3 < 1
  - u(R, R) = 1

- Conclusion: Not all games have an ESS!