Game Theory
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Lecture 3

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Lecture 2 recap

- Defined Pareto optimality
  - Coordination games
- Studied games with continuous action space
  - Always have a Nash equilibrium with some conditions
  - Cournot duopoly example

→ Can we always find a Nash equilibrium for all games?

→ How?

Outline

1. Mixed strategies
    - Best response and Nash equilibrium
2. Mixed strategies Nash equilibrium computation
3. Interpretations of mixed strategies

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Example: installing checkpoints

- Two road, Police choose on which to check, Terrorists choose on which to pass

		Terrorist	
		R1	R2
Police	R1	1 , -1	-1, 1
	R2	-1, 1	1, -1

- Can you find a Nash equilibrium?

→ Players must **randomize**

Matching pennies

Player 2

		heads	tails
Player 1	heads	1 , -1	-1, 1
	tails	-1, 1	1, -1

- Similar examples:
  - Checkpoint placement
  - Intrusion detection
  - Penalty kick
  - Tennis game
- Need to be unpredictable

Pure strategies/Mixed strategies

- Game $\left(N,(A_i)_{i\in N},(u_i)_{i\in N}\right)$
- A$_i$: set of actions of player i (what we called S$_i$ before)
- Action = **pure strategy**
- **Mixed strategy**: distribution over pure strategies

$$s_i \in S_i = \Delta(A_i)$$

  - Include pure strategy as special case
  - Support: $\text{supp } s_i = \{a_i \in A_i : s_i(a_i) > 0\}$
- Strategy profile: $s = (s_1, \cdots, s_n) \in S = S_1 \times \cdots \times S_n$

Matching pennies: payoffs

		Player 2	
		heads	tails
Player 1	heads	1 , -1	-1, 1
	tails	-1, 1	1, -1

- What is Player 1’s payoff if Player 2 plays s_2 = (1/4, 3/4) and he plays:
  - Heads?
  - Tails?
  - s_1 = (½, ½)?

Payoffs in mixed strategies: general formula

- Game $\left(N, (A_i)_{i\in N}, (u_i)_{i\in N}\right)$, let $A = \underset{i\in N}{\times} A_i$
- If players follow a mixed-strategy profile s, the expected payoff of player i is:

$$u_i(s) = \sum_{a\in A} u_i(a)\Pr(a \mid s) \quad \text{where} \quad \Pr(a \mid s) = \prod_{i\in N} s_i(a_i)$$

- a: pure strategy (or action) profile
- Pr(a|s): probability of seeing a given the mixed strategy profile s

Matching pennies: payoffs check

		Player 2	
		heads	tails
Player 1	heads	1 , -1	-1, 1
	tails	-1, 1	1, -1

- What are the payoffs of Player 1 and Player 2 if s = ((½, ½), (¼, ¾))?
- Does that look like it could be a Nash equilibrium?

Best response

- The definition for mixed strategies is unchanged!

Definition: Best Response

Player *i*'s strategy $\hat{s}_i$ is a BR to strategy s_{-i} of other players if:

$$u_i(\hat{s}_i , s_{-i}) \geq u_i(s'_i , s_{-i}) \text{ for all } s'_i \text{ in } S_i$$

- $BR_i(s_{-i})$: set of best responses of i to s_{-i}

Matching pennies: best response

		Player 2	
		heads	tails
Player 1	heads	1 , -1	-1, 1
	tails	-1, 1	1, -1

- What is the best response of Player 1 to s_2 = (¼, ¾)?

- For all s_1, $u_1(s_1, s_2)$ lie between $u_1(\text{heads}, s_2)$ and $u_1(\text{tails}, s_2)$ (the weighted average lies between the pure strategies exp. Payoffs)

→ Best response is tails!

Important property

- If a mixed strategy is a best response then each of the pure strategies in the mix must be best responses

➔ They must yield the same expected payoff

Proposition:

For any (mixed) strategy s_{-i}, if $s_i \in BR_i(s_{-i})$, then $a_i \in BR_i(s_{-i})$ for all a_i such that $s_i(a_i) > 0$.

In particular, $u_i(a_i, s_{-i})$ is the same for all a_i such that $s_i(a_i) > 0$

Wordy proof

- Suppose it were not true. Then there must be at least one pure strategy a_i that is assigned positive probability by my best-response mix and that yields a lower expected payoff against s_i
- If there is more than one, focus on the one that yields the lowest expected payoff. Suppose I drop that (low-yield) pure strategy from my mix, assigning the weight I used to give it to one of the other (higher-yield) strategies in the mix
- This must raise my expected payoff
- But then the original mixed strategy cannot have been a best response: it does not do as well as the new mixed strategy
- This is a contradiction

Matching pennies again

Player 2

		heads	tails
Player 1	heads	1 , -1	-1, 1
	tails	-1, 1	1, -1

- What is the best response of Player 1 to s_2 = (¼, ¾)?
- What is the best response of Player 1 to s_2 = (½, ½)?

Nash equilibrium definition

Definition: Nash Equilibrium

A strategy profile (s_1*, s_2*,…, s_N*) is a Nash Equilibrium (NE) if, for each i, her choice s_i* is a best response to the other players' choices s_{-i}*

- Same definition as for pure strategies!
  - But here the strategies s_i^* are mixed strategies

Matching pennies again

		Player 2	
		heads	tails
Player 1	heads	1 , -1	-1, 1
	tails	-1, 1	1, -1

- Nash equilibrium: ((½, ½), (½, ½))

Nash equilibrium existence theorem

Theorem: **Nash (1951)**

Every finite game has a Nash equilibrium.

- In mixed strategy!
  - Not true in pure strategy
- Finite game: finite set of player and finite action set for all players
  - Both are necessary!
- Proof: reduction to Kakutani's fixed-point thm

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1. Mixed strategies
   - Best response and Nash equilibrium
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Computation of mixed strategy NE

- Hard if the support is not known
- If you can guess the support, it becomes very easy, using the property shown earlier:

Proposition:

For any (mixed) strategy s_{-i}, if $s_i \in BR_i(s_{-i})$, then $a_i \in BR_i(s_{-i})$ for all a_i such that $s_i(a_i) > 0$.

In particular, $u_i(a_i, s_{-i})$ is the same for all a_i such that $s_i(a_i) > 0$ (i.e., a_i in the support of s_i)

Example: battle of the sexes

		Player 2	
		Opera	Soccer
Player 1	Opera	**2,1**	**0,0**
	Soccer	0,0	1,2

- We have seen that (O, O) and (S, S) are NE

- Is there any other NE (in mixed strategies)?
  - Let’s try to find a NE with support {O, S} for each player

Example: battle of the sexes (2)

		Player 2	
		Opera	Soccer
Player 1	Opera	**2,1**	**0,0**
	Soccer	0,0	1,2

- Let $s_2 = (p, 1-p)$
- If s_1 is a BR with support {O, S}, then Player 1 must be indifferent between O and S

→ $p = 1/3$

Example: battle of the sexes (3)

		Player 2	
		Opera	Soccer
Player 1	Opera	2,1	0,0
	Soccer	0,0	1,2

- Similarly, let $s_1 = (q, 1\text{-}q)$
- If s_2 is a BR with support {O, S}, then Player 2 must be indifferent between O and S

→ $q = 2/3$

Example: battle of the sexes (4)

		Player 2	
		Opera	Soccer
Player 1	Opera	**2,1**	**0,0**
	Soccer	0,0	1,2

- Conclusion: ((2/3, 1/3), (1/3, 2/3)) is a NE

Example: prisoner’s dilemma

- We know that (D, D) is NE
- Can we find a NE with support {C, D} with each?

		Prisoner 2	
		D	C
Prisoner 1	D	-5, -5	0, -6
	C	-6, 0	-2, -2

- A NE in strictly dominant strategies is unique!

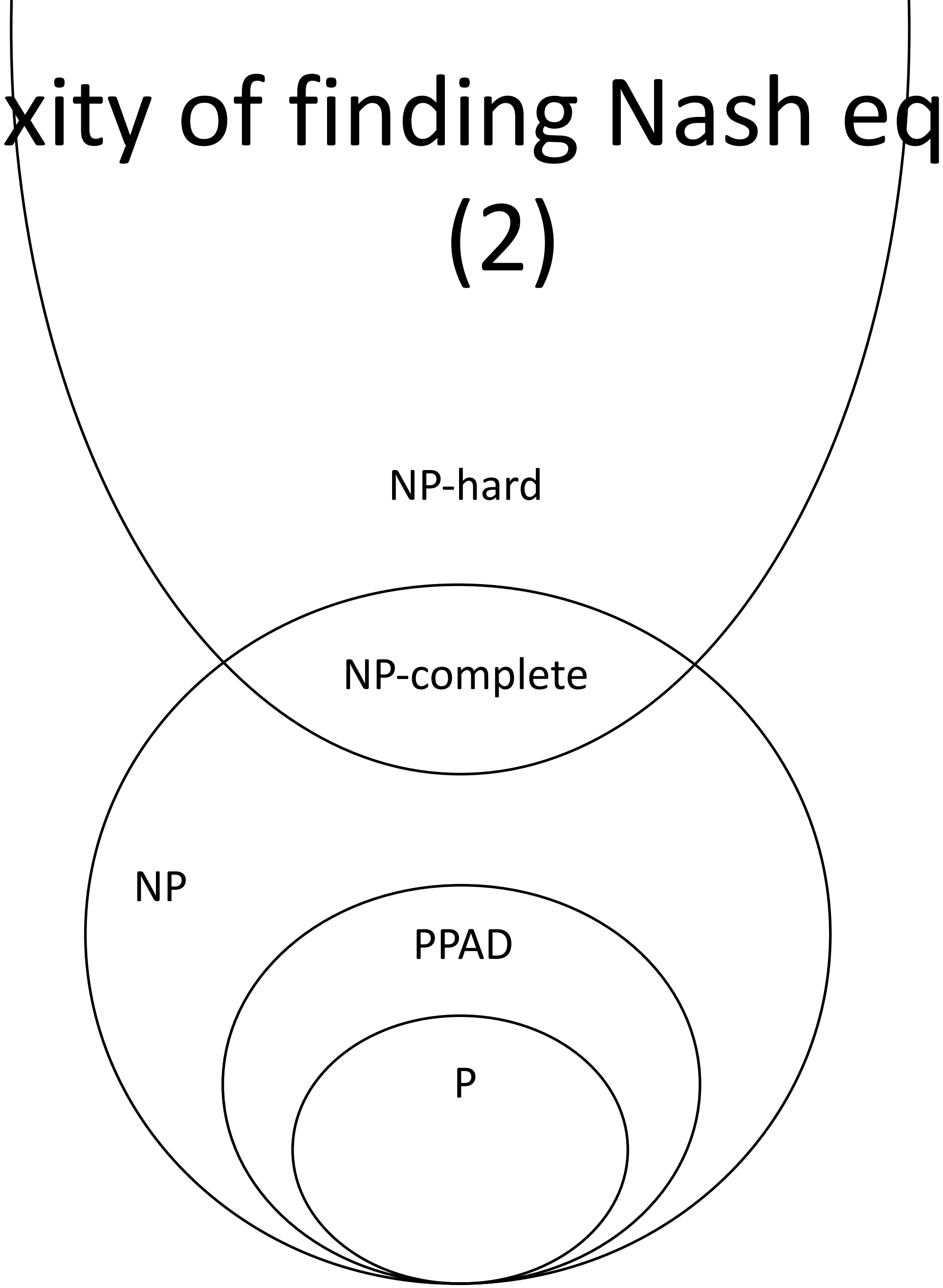
General methods to compute Nash equilibrium

- If you know the support, write the equations translating indifference between strategies in the support (works for any number of actions!)
- Otherwise:
  - The Lemke-Howson Algorithm (1964)
  - Support enumeration method (Porter et al. 2004)
    - Smart heuristic search through all sets of support
- Exponential time worst case complexity

Complexity of finding Nash equilibrium

- Is it NP-complete?
  - No, we know there is a solution
  - But many derived problems are (e.g., does there exists a strictly Pareto optimal Nash equilibrium?)
- PPAD ("Polynomial Parity Arguments on Directed graphs") [Papadimitriou 1994]
- Theorem: Computing a Nash equilibrium is PPAD-complete [Chen, Deng 2006]

Complexity of finding Nash equilibrium (2)



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Mixed strategies interpretations

- Players randomize
- Belief of others’ actions (that make you indifferent)
- Empirical frequency of play in repeated interactions
- Fraction of a population
  - Let’s see an example of this one

The Income Tax Game (1)

		Tax payer		
		Honest	Cheat	
Auditor	A	**2,0**	**4,-10**	p
	N	4,0	0,4	(1-p)
		q	1-q	

- Assume simultaneous move game
- Is there a pure strategy NE?
- Find mixed strategy NE

The Income Tax Game: NE computation

- Mixed strategies NE:

$$\left.\begin{array}{l} E[U_1(A,(q,1-q))] = 2q + 4(1-q) \\ E[U_1(N,(q,1-q))] = 4q + 0(1-q) \end{array}\right\} 2q = 4(1-q) \Rightarrow q = \frac{2}{3}$$

$$\left.\begin{array}{l} E[U_2(H,(p,1-p))] = 0 \\ E[U_2(C,(p,1-p))] = -10p + 4(1-p) \end{array}\right\} 4 = 14p \Rightarrow p = \frac{2}{7}$$

Look at tax payers payoffs

To find auditors mixing

The Income Tax Game: mixed strategy interpretation

- From the auditor’s point of view, he/she is going to audit a single tax payer 2/7 of the time

➔This is actually a **randomization** (which is applied by law)

- From the tax payer perspective, he/she is going to be honest 2/3 of the time

➔ This in reality implies that 2/3rd of population is going to pay taxes honestly, i.e., this is a **fraction of a large population** paying taxes

The Income Tax Game (6)

- What could ever be done if one policy maker (e.g. the government) would like to increase the proportion of honest tax payers?

- One idea could be for example to “prevent” fraud by increasing the number of years a tax payer would spend in jail if found guilty

The Income Tax Game: Trying to make people pay

		Tax payer		
		Honest	Cheat	
Auditor	A	**2,0**	**4,-20**	p
	N	4,0	0,4	(1-p)
		q	1-q	

- How to make people pay their taxes?
- One idea: increase penalty for cheating
- What is the new equilibrium?

The Income Tax Game: new NE

$$\left.\begin{aligned} E\left[U_1\left(A,\left(q,1-q\right)\right)\right] &= 2q+4(1-q) \\ E\left[U_1\left(N,\left(q,1-q\right)\right)\right] &= 4q+0(1-q) \end{aligned}\right\} 2q = 4(1-q) \Rightarrow \boxed{q = \frac{2}{3}}$$

$$\left.\begin{aligned} E\left[U_2\left(H,\left(p,1-p\right)\right)\right] &= 0 \\ E\left[U_2\left(C,\left(p,1-p\right)\right)\right] &= -20p+4(1-p) \end{aligned}\right\} 24p = 4 \Rightarrow \boxed{p = \frac{1}{6} <}\frac{2}{7}$$

- The proportion of honest tax payers didn’t change!
  - What determines the equilibrium mix for the column player is the row player’s payoffs
- The probability of audit decreased
  - Still good, audits are expensive
- To make people pay tax: change auditor’s payoff
  - Make audits cheaper, more profitable

Important remark

- **Row player’s NE mix determined by column player’s payoff and vice versa**
- Neutralize the opponent (make him indifferent)
- In some sense the opposite of optimization (my choice is independent of my own payoff)

The penalty kick game

- 2 players: kicker and goalkeeper
- 2 actions each
  - Kicker: kick left, kick right
  - Goalkeeper: jump left, jump right
- Payoff: probability to score for the kicker, probability to stop it for the goalkeeper
- Scoring probabilities:

		Goal keeper	
		L	R
Kicker	L	**58.30**	**94.97**
	R	92.91	69.92

The penalty kick game: results

- Ignacio Palacios-Huerta. Professionals Play Minimax. Review of Economics Studies (2003).
- Result:

	Goal L	Goal R	Kicker L	Kicker R
NE prediction	**41.99**	**58.01**	**38.54**	**61.46**
Observed freq.	42.31	57.69	39.98	60.02

- For a given kicker, his strategy is also serially independent

Summary

- Mixed strategies: distribution over actions
  - A Nash equilibrium in mixed strategies always exists for finite games
  - Computation is easy if the support is known
    - All pure strategies in the support of a best response are also best responses
    - Makes other player indifferent in his support
  - Computation is hard if the support is not known
  - Several interpretations depending on the game at stake