Game Theory
--
Lecture 2

Patrick Loiseau
EURECOM
Fall 2016

Lecture 1 recap

- Defined games in normal form
- Defined dominance notion
  - Iterative deletion
  - Does not always give a solution
- Defined best response and Nash equilibrium
  - Computed Nash equilibrium in some examples

→ Are some Nash equilibria better than others?

→ Can we always find a Nash equilibrium?

Outline

1. Coordination games and Pareto optimality
2. Games with continuous action sets
   – Equilibrium computation and existence theorem
   – Example: Cournot duopoly

Outline

1. Coordination games and Pareto optimality
2. Games with continuous action sets
   - Equilibrium computation and existence theorem
   - Example: Cournot duopoly

The Investment Game

- The players: you
- The strategies: each of you chooses between investing nothing in a class project ($0) or investing ($10)
- Payoffs:
  - If you don't invest your payoff is $0
  - If you invest you make a **net profit** of $5 (gross profit = $15; investment $10) if more than 90% of the class chooses to invest. Otherwise, you lose $10

- Choose your action (no communication!)

Nash equilibrium

- What are the Nash equilibria?

- Remark: to find Nash equilibria, we used a "guess and check method"
  - Checking is easy, guessing can be hard

The Investment Game again

- Recall that:
  - Players: you
  - Strategies: invest $0 or invest $10
  - Payoffs:
    - If no invest → $0
    - If invest $10 → { $5 net profit if ≥ 90% invest ; -$10 net profit if < 90% invest }

- Let's play again! (no communication)

- We are heading toward an equilibrium

➔ There are certain cases in which playing converges in a natural sense to an equilibrium

Pareto domination

- Is one equilibrium better than the other?

Definition: Pareto domination

A strategy profile s Pareto dominates strategy profile s' iif for all i, $u_i(s) \geq u_i(s')$ and there exists j such that $u_j(s) > u_j(s')$;
i.e., all players have at least as high payoffs and at least one player has strictly higher payoff.

- In the investment game?

Convergence to equilibrium in the Investment Game

- Why did we converge to the wrong NE?
- Remember when we started playing
  - We were more or less 50 % investing
- The starting point was already bad for the people who invested for them to lose confidence
- Then we just tumbled down

- What would have happened if we started with 95% of the class investing?

Coordination game

- This is a ***coordination game***
  - We'd like everyone to coordinate their actions and <u>invest</u>
- Many other examples of coordination games
  - Party in a Villa
  - On-line Web Sites
  - Establishment of technological monopolies (Microsoft, HDTV)
  - Bank runs
- Unlike in prisoner's dilemma, <u>***communication helps***</u> in coordination games → <u>***scope for leadership***</u>
  - In prisoner's dilemma, a trusted third party (TTP) would need to impose players to adopt a strictly dominated strategy
  - In coordination games, a TTP just leads the crowd towards a better NE point (there is no dominated strategy)

Battle of the sexes

		Player 2	
		Opera	Soccer
Player 1	Opera	**2,1**	**0,0**
	Soccer	0,0	1,2

- Find the NE
- Is there a NE better than the other(s)?

Coordination Games

- Pure coordination games: there is no conflict whether one NE is better than the other
  - E.g.: in the investment game, we all agreed that the NE with everyone investing was a "better" NE

- General coordination games: there is a source of conflict as players would agree to coordinate, but one NE is "better" for a player and not for the other
  - E.g.: Battle of the Sexes

➔ Communication might fail in this case

Pareto optimality

Definition: **Pareto optimality**
A strategy profile s is Pareto optimal if there does not exist a strategy profile s’ that Pareto dominates s.

- Battle of the sexes?

Outline

1. Coordination games and Pareto optimality
2. Games with continuous action sets
   – Equilibrium computation and existence theorem
   – Example: Cournot duopoly

The partnership game (see exercise sheet 2)

- Two partners choose effort s_i in $S_i=[0, 4]$
- Share revenue and have quadratic costs

$$u_1(s_1 , s_2) = ½ [4 (s_1 + s_2 + b\, s_1 s_2)] - s_1^2$$

$$u_2(s_1 , s_2) = ½ [4 (s_1 + s_2 + b\, s_1 s_2)] - s_2^2$$

- Best responses:

$$\hat{s}_1 = 1 + b\, s_2 \quad = BR_1(s_2)$$

$$\hat{s}_2 = 1 + b\, s_1 \quad = BR_2(s_1)$$

Finding the best response (with twice continuously differentiable utilities)

$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} = 0$$

- First order condition (FOC)

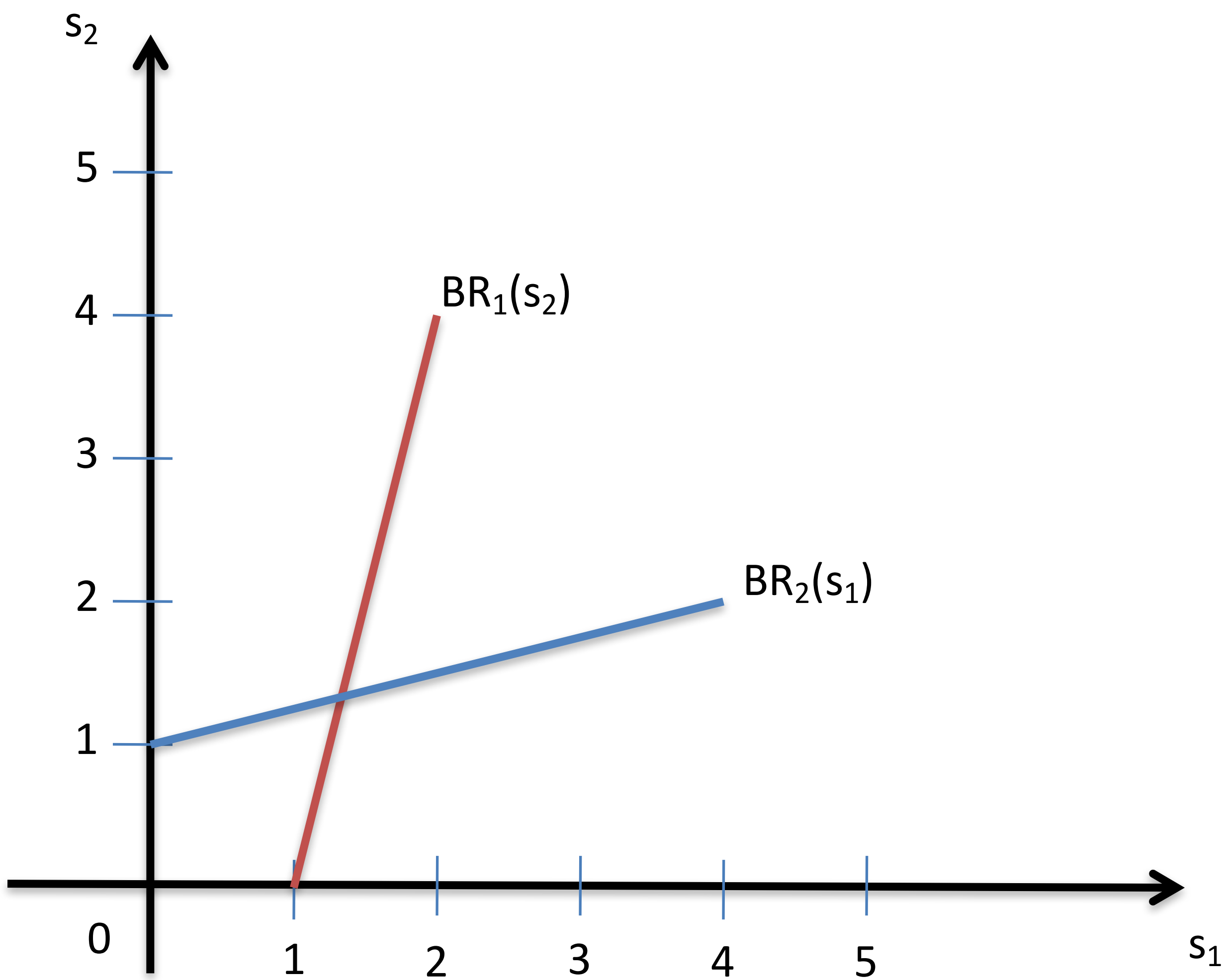
$$\frac{\partial^2 u_1(s_1, s_2)}{\partial^2 s_1} \leq 0$$

- Second order condition (SOC)

- Remark: the SOC is automatically satisfied if $u_i(s_i, s_{-i})$ is concave in s_i for all s_{-i} (very standard assumption)
- Remark 2: be careful with the borders!
  - Example $u_1(s_1, s_2) = 10-(s_1+s_2)^2$
  - $S_1=[0, 4]$, what is the BR to $s_2=2$?
  - Solving the FOC, what do we get?

  - When the FOC solution is outside S_i, the BR is at the border

Nash equilibrium graphically



- NE is fixed point of $(s_1, s_2) \rightarrow (BR(s_2), BR(s_1))$

Best response correspondence

- Definition: $\hat{s}_i$ is a BR to s_{-i} if $\hat{s}_i$ solves **max** $u_i(s_i , s_{-i})$
- The BR to s_{-i} may not be unique!
- $BR(s_{-i})$: set of s_i that solve **max** $u_i(s_i , s_{-i})$
- The definition can be written:
  $\hat{s}_i$ is a BR to s_{-i} if $\hat{s}_i \in BR_i(s_{-i}) = \underset{s_i}{\operatorname{argmax}}\, u_i(s_i, s_{-i})$

- Best response correspondence of i: $s_{-i} \rightarrow BR_i(s_{-i})$
- (Correspondence = set-valued function)

Nash equilibrium as a fixed point

- Game $\left(N, (S_i)_{i\in N}, (u_i)_{i\in N}\right)$
- Let's define $S = \times_{i\in N} S_i$ (set of strategy profiles) and the correspondence

$$B : S \to S$$

$$s \mapsto B(s) = \times_{i\in N} BR_i(s_{-i})$$

- For a given s, B(s) is the set of strategy profiles s' such that s_i' is a BR to s_{-i} for all i.
- A strategy profile s^* is a Nash eq. iif $s^* \in B(s^*)$ (just a re-writing of the definition)

Kakutani's fixed point theorem

Theorem: Kakutani's fixed point theorem

Let *X* be a compact convex subset of R^n and let $f : X \to X$ be a set-valued function for which:

- for all $x \in X$, the set $f(x)$ is nonempty convex;
- the graph of f is closed.

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$

Closed graph (upper hemicontinuity)

- Definition: f has closed graph if for all sequences (x_n) and (y_n) such that y_n is in $f(x_n)$ for all n, $x_n \rightarrow x$ and $y_n \rightarrow y$, y is in f(x)
- Alternative definition: f has closed graph if for all x we have the following property: for any open neighborhood V of f(x), there exists a neighborhood U of x such that for all x in U, f(x) is a subset of V.
- Examples:

Existence of (pure strategy) Nash equilibrium

Theorem: Existence of pure strategy NE

Suppose that the game $\left(N,(S_i)_{i\in N},(u_i)_{i\in N}\right)$ satisfies:

- The action set S_i of each player is a nonempty compact convex subset of R^n
- The utility u_i of each player is continuous in s (on S) and concave in s_i (on S_i)

Then, there exists a (pure strategy) Nash equilibrium.

- Remark: the concave assumption can be relaxed

Proof

- Define B as before. B satisfies the assumptions of Kakutani's fixed point theorem
- Therefore B has a fixed point which by definition is a Nash equilibrium!

- Now, we need to actually verify that B satisfies the assumptions of Kakutani's fixed point theorem!

Example: the partnership game

- N = {1, 2}
- S = [0,4]x[0,4] compact convex
- Utilities are continuous and concave

$$u_1(s_1 , s_2) = ½ [4 (s_1 + s_2 + b\, s_1\, s_2)] - s_1^2$$
$$u_2(s_1 , s_2) = ½ [4 (s_1 + s_2 + b\, s_1\, s_2)] - s_2^2$$

- Conclusion: there exists a NE!

- Ok, for this game, we already knew it!
- But the theorem is much more general and applies to games where finding the equilibrium is much more difficult

One more word on the partnership game before we move on

- We have found (see exercises) that
  - At Nash equilibrium:

$$s^*_1 = s^*_2 = 1/(1-b)$$

  - To maximize the sum of utilities:

$$s^W_1 = s^W_2 = 1/(1/2-b) > s^*_1$$

- Sum of utilities called social welfare
- Both partners would be better off if they worked s^W_1 (with social planner, contract)
- Why do they work less than efficient?

Externality

- At the margin, I bear the cost for the extra unit of effort I contribute, but I'm only reaping half of the induced profits, because of **profit sharing**

- This is known as an "externality"

➔ When I'm figuring out the effort I have to put I don't take into account that other half of profit that goes to my partner

➔ In other words, my effort benefits my partner, not just me

- Externalities are omnipresent: public good problems, free riding, etc. (see more in the netecon course)

Outline

1. Coordination games and Pareto optimality
2. Games with continuous action sets
   - Equilibrium computation and existence theorem
   - Example: Cournot duopoly

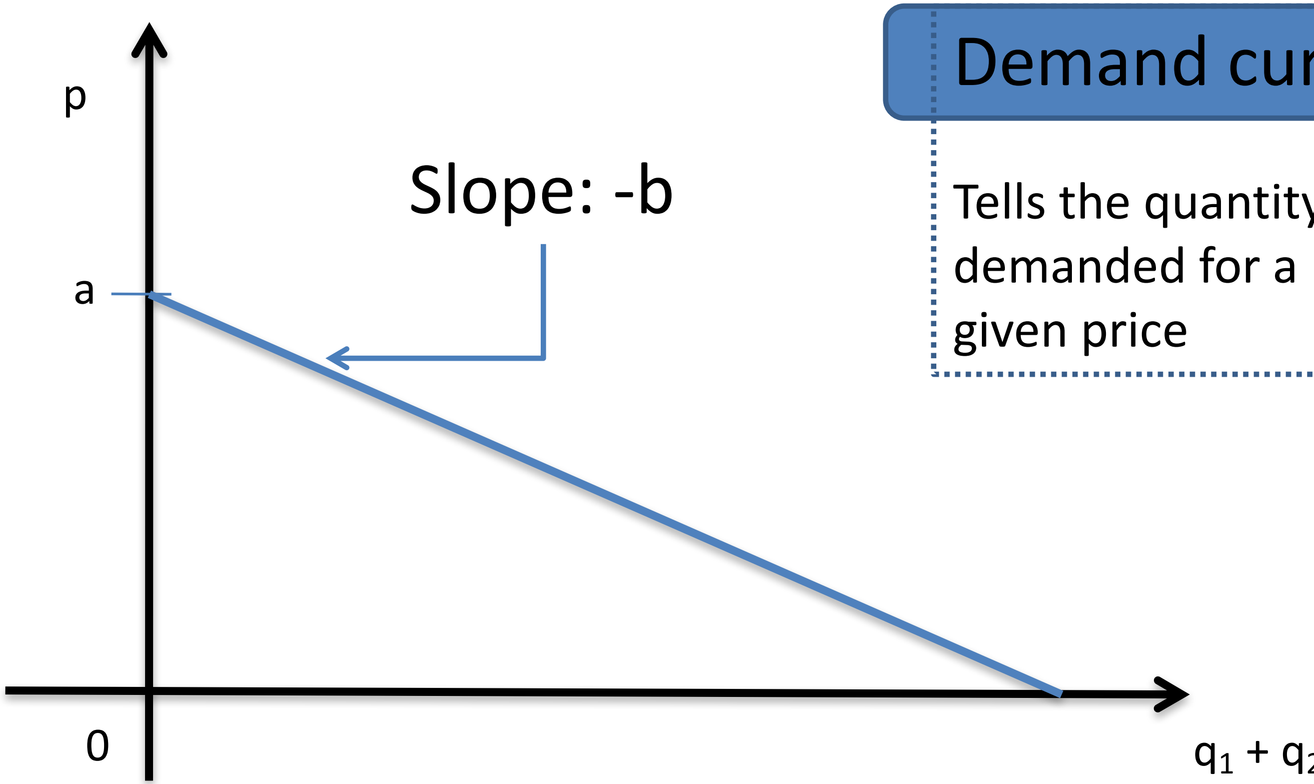
Cournot Duopoly

- Example of application of games with continuous action set
- This game lies between two extreme cases in economics, in situations where firms (e.g. two companies) are competing on the same market
  - Perfect competition
  - Monopoly
- We're interested in understanding what happens in the middle
  - The game analysis will give us interesting economic insights on the duopoly market

Cournot Duopoly: the game

- The players: 2 Firms, e.g., Coke and Pepsi
- Strategies: quantities players produce of ***identical*** products: q_i, q_{-i}
  - Products are perfect substitutes
- Cost of production: c * q
  - Simple model of ***constant marginal cost***
- Prices: $p = a - b(q_1 + q_2) = a - bQ$
  - Market-clearing price

Price in the Cournot duopoly



Demand curve

Tells the quantity demanded for a given price

Cournot Duopoly: payoffs

- The payoffs: firms aim to ***maximize profit***

$$u_1(q_1,q_2) = p * q_1 - c * q_1$$
$$p = a - b\,(q_1 + q_2)$$

➢ $u_1(q_1,q_2) = a * q_1 - b * q^2{}_1 - b * q_1\, q_2 - c * q_1$

- The game is symmetric

➢ $u_2(q_1,q_2) = a * q_2 - b * q^2{}_2 - b * q_1\, q_2 - c * q_2$

Cournot Duopoly: best responses

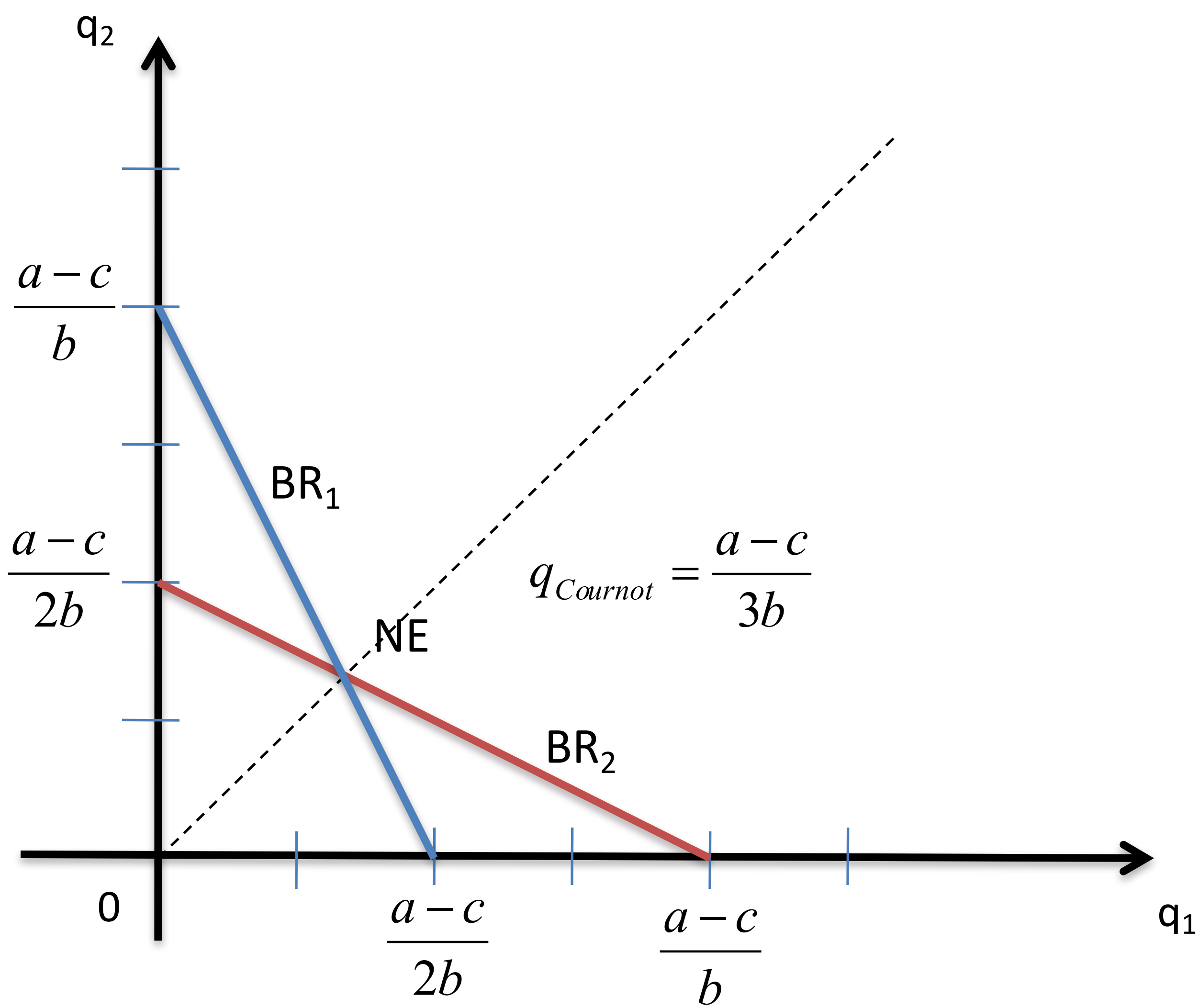
- First order condition $a - 2bq_1 - bq_2 - c = 0$
- Second order condition $-2b < 0$

  [make sure it's a max]

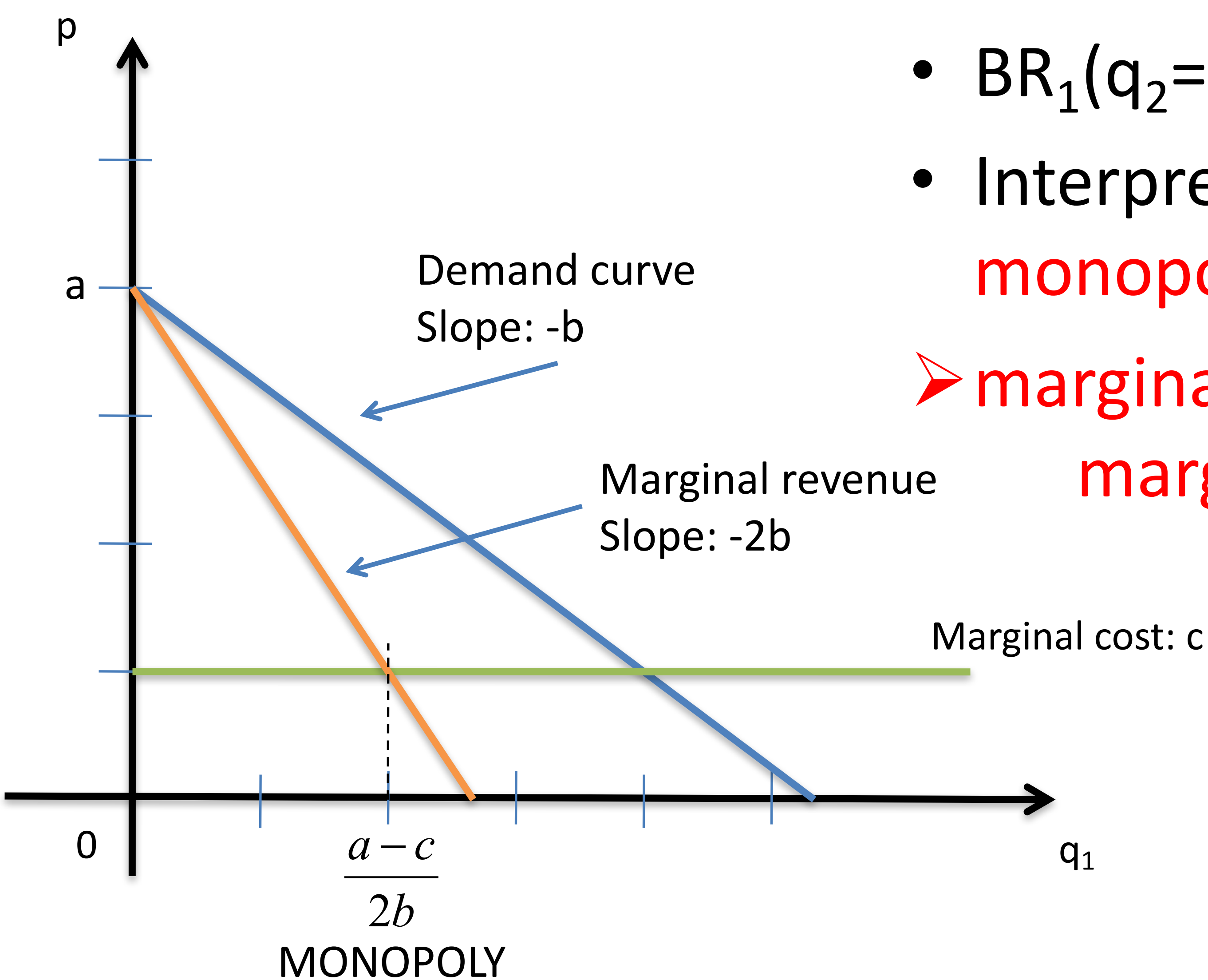
➔

$$\begin{cases} \hat{q}_1 = BR_1(q_2) = \dfrac{a-c}{2b} - \dfrac{q_2}{2} \\ \hat{q}_2 = BR_2(q_1) = \dfrac{a-c}{2b} - \dfrac{q_1}{2} \end{cases}$$

Cournot Duopoly: best response diagram and Nash equilibrium

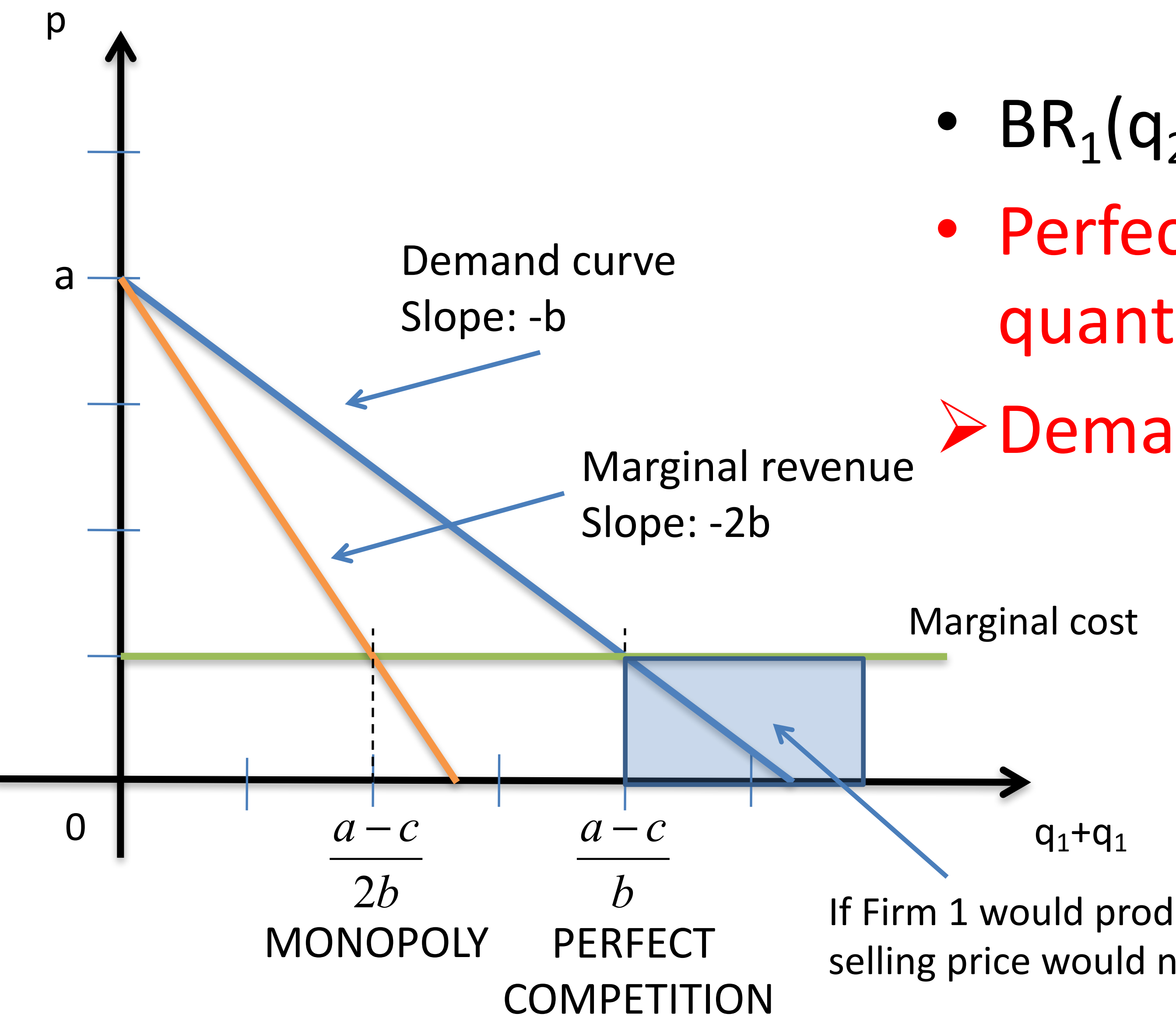


Best response at q_2=0



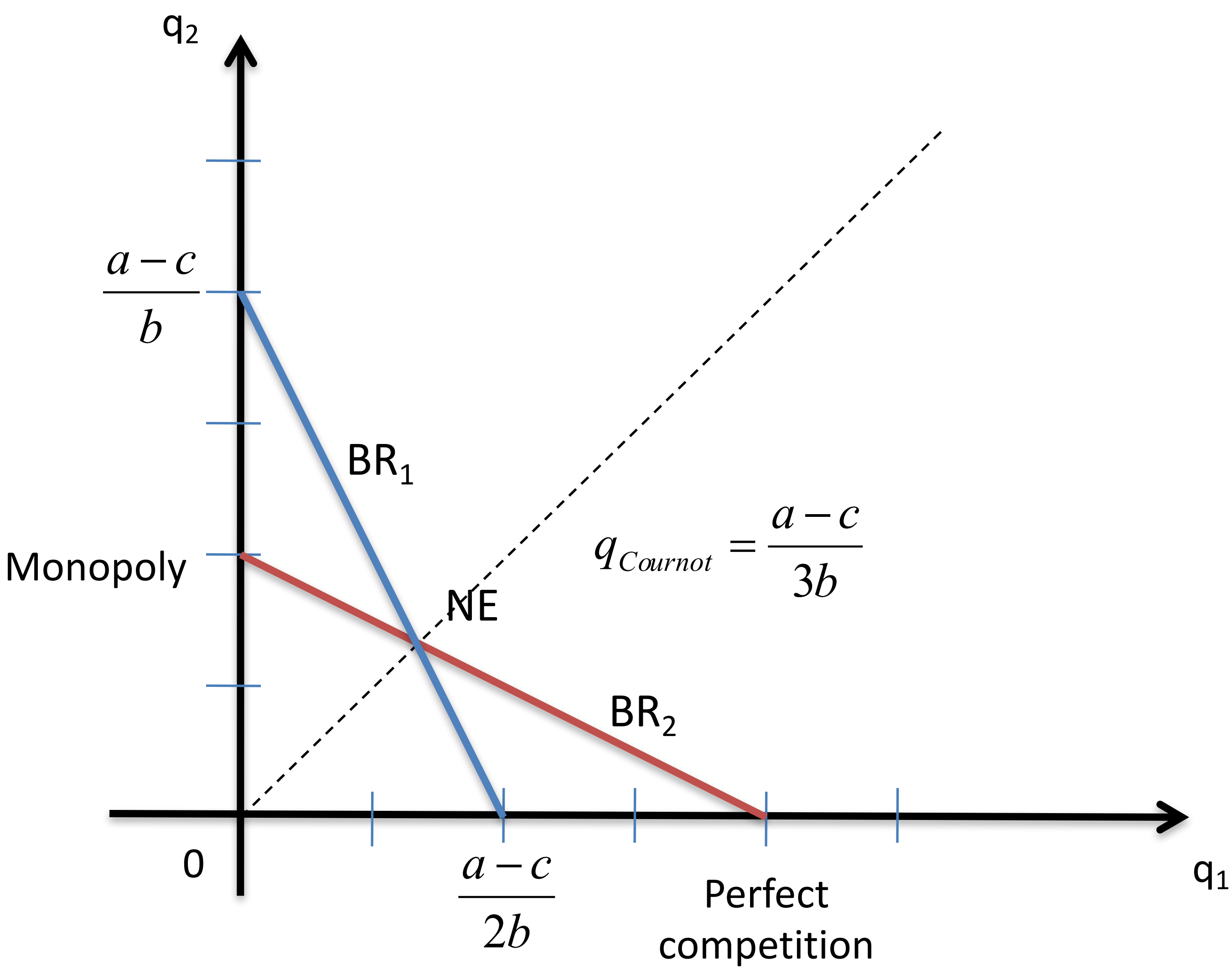
- $BR_1(q_2=0) = (a-c)/(2b)$
- Interpretation: monopoly quantity
  - marginal revenue = marginal cost

When is $BR_1(q_2) = 0$?



- $BR_1(q_2=(a-c)/b) = 0$
- Perfect competition quantity
  - ➢ Demand = marginal cost

Cournot Duopoly: best response diagram and Nash equilibrium



Strategic substitutes/complements

- In Cournot duopoly: the more the other player does, the less I would do

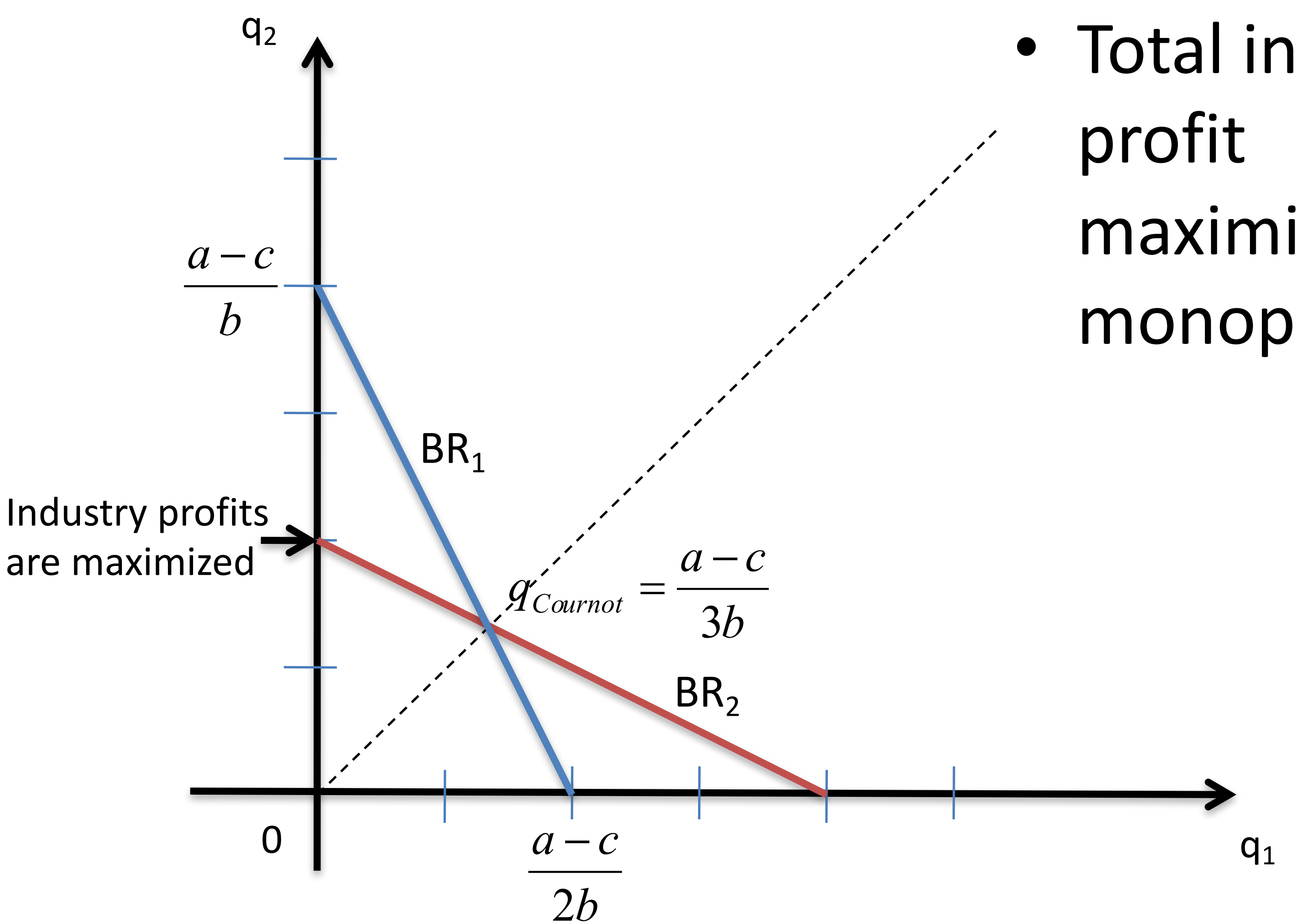
➔ This is a game of ***strategic substitutes***

  - Note: of course the goods were substitutes
  - We're talking about strategies here

- In the partnership game, it was the opposite: the more the other player would the more I would do

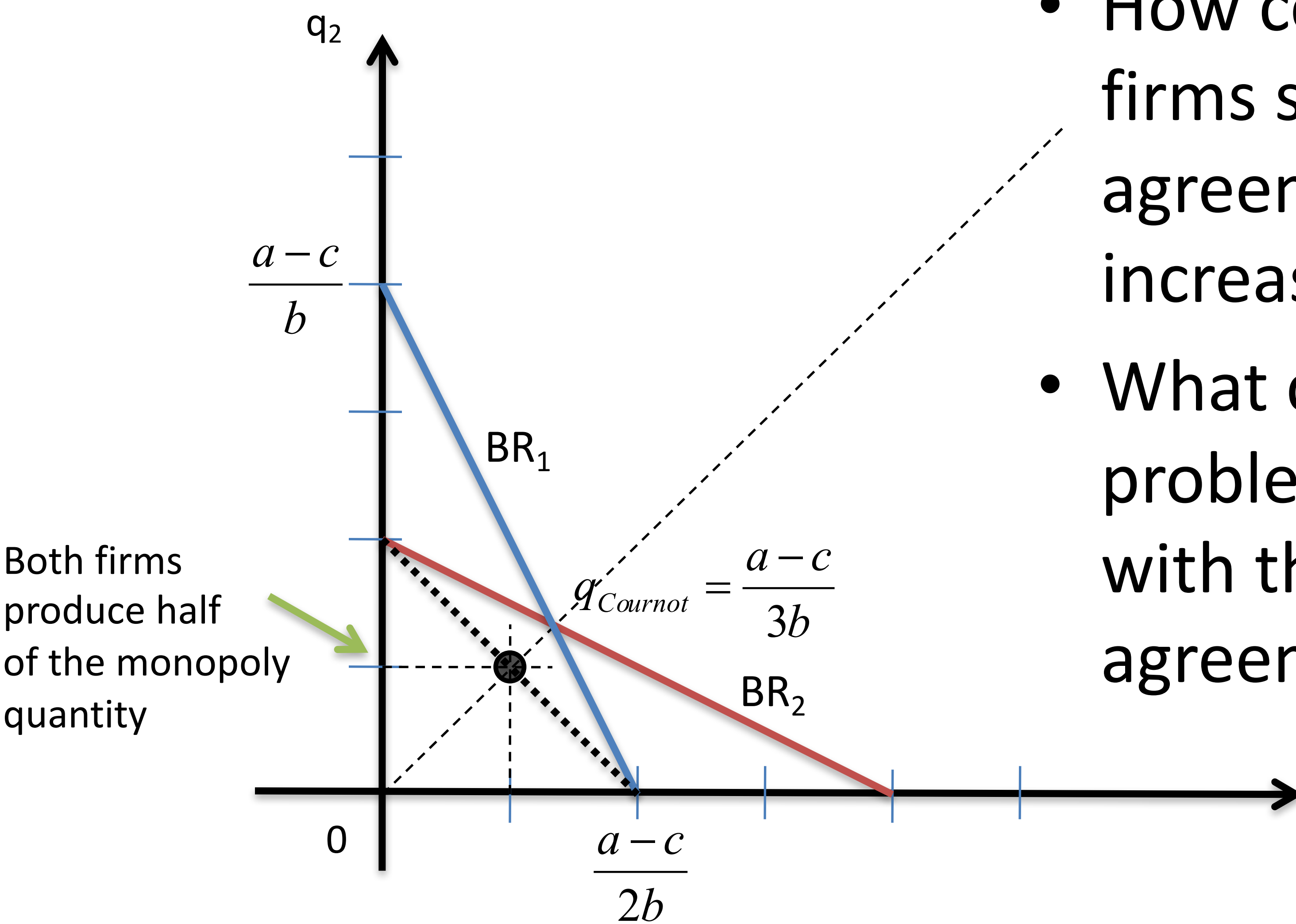
➔ This is a game of ***strategic complements***

Cournot duopoly: Market perspective



- Total industry profit maximized for monopoly

Cartel, agreement



- How could the firms set an agreement to increase profit?
- What can the problems be with this agreement ?

Cournot Duopoly: last observations

- How do quantities and prices we've encountered so far compare?

QUANTITIES

$$\frac{a-c}{b} \qquad \frac{2(a-c)}{3b} \qquad \frac{a-c}{2b}$$

PRICES

Summary

- Coordination games
  - Pareto optimal NE sometimes exist
  - Scope for communication / leadership
- Games with continuous action sets (pure strategies)
  - Compute equilibrium with FOC, SOC
  - Equilibrium exists under concavity and continuity conditions
  - Cournot duopoly