Game Theory
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Lecture 1

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Lecture 1 outline

1. Introduction
2. Definitions and notation
   - Game in normal form
   - Strict and weak dominance
3. Iterative deletion of dominated strategy
   - A first model in politics
4. Best response and Nash equilibrium

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Let’s play the “grade game”

Without showing your neighbors what you are doing, write down on a form either the letter ***alpha*** *or the letter* ***beta****. Think of this as a “grade bid”. I will randomly pair your form with one other form. Neither you nor your pair will ever know with whom you were paired. Here is how grades may be assigned for this class:*

- If you put alpha and your pair puts beta, then you will get grade A, and your pair grade C;
- If both you and your pair put alpha, then you both will get the grade B-;
- If you put beta and your pair puts alpha, then you will get the grade C and your pair grade A;
- If both you and your pair put beta, then you will both get grade B+

What is game theory?

- Game theory is a method of studying **strategic** situations, i.e., where the outcomes that affect you depend on actions of others, not only yours
- Informally:
  - At one end we have Firms in perfect competition: in this case, firms are price takers and do not care about what other do
  - At the other end we have Monopolist Firms: in this case, a firm doesn't have competitors to worry about, they're not price-takers but they take the demand curve
  - Everything in between is strategic, i.e., everything that constitutes imperfect competition
    - Example: The automotive industry
- Game theory has become a multidisciplinary area
  - Economics, mathematics, computer science, engineering…

Outcome matrix

- Just reading the text is hard to absorb, let's use a concise way of representing the game:

my pair

me		alpha	beta
	alpha	B -	A
	beta	C	B +

my grades

my pair

me		alpha	beta
	alpha	B -	C
	beta	A	B +

pair's grades

Outcome matrix (2)

- We use a more compact representation:

my pair

		alpha	beta
me	alpha	B - , B -	A , C
	beta	C , A	B + , B +

1st grade: row player (my grade)

2nd grade: column player (my pair's grade)

This is an **outcome** matrix:

It tells us everything that was in the game we saw

The grade game: discussion

- What did you choose? Why?
- Two possible way of thinking:
  - Regardless of my partner choice, there would be better outcomes for me by choosing alpha rather than beta;
  - We could all be collusive and work together, hence by choosing beta we would get higher grades.
- We don’t have a game yet!
  - We have **players** and **strategies** (i.e., possible actions)
  - We are missing **objectives**
- Objectives can be defined in two ways
  - Preferences, i.e., ordering of possible outcomes
  - Payoffs or utility functions

The grade game: payoff matrix

- Possible payoffs: in this case we only care about our own grades

# of utiles, or utility:

(A,C) → 3

(B-, B-) → 0

Hence the preference order is:

A > B+ > B- > C

		my pair: alpha	my pair: beta
me	alpha	0 , 0	3, -1
me	beta	-1, 3	1,1

- How to choose an action here?

Strictly dominated strategies

- Play alpha!
  - Indeed, no matter what the pair does, by playing alpha you would obtain a higher payoff

Definition:
We say that my strategy alpha **strictly dominates** my strategy beta, if my payoff from alpha is **strictly greater** than that from beta, regardless of what others do.

→ **Do not play a strictly dominated strategy!**

Rational choice outcome

- If we (me and my pair) reason selfishly, we will both select alpha, and get a payoff of 0;
- But we could end up both with a payoff of 1…
- What's the problem with this?
  - Suppose you have super mental power and oblige your partner to agree with you and choose beta, so that you both would end up with a payoff of 1…
  - Even with **communication**, it wouldn't work, because at this point, you'd be better of by choosing alpha, and get a payoff of 3

→ ***Rational choice (i.e., not choosing a dominated strategy) can lead to bad outcomes!***

- Solutions?
  - Contracts, treaties, regulations: change payoff
  - Repeated play

The prisoner's dilemma

- Important class of games
- Other examples
  1. Joint project:
     - Each individual may have an incentive to shirk
  2. Price competition
     - Each firm has an incentive to undercut prices
     - If all firms behave this way, prices are driven down towards marginal cost and industry profit will suffer
  3. Common resource
     - Carbon emissions
     - Fishing

		Prisoner 2: D	Prisoner 2: C
Prisoner 1	D	-5, -5	0, -6
	C	-6, 0	-2, -2

Another possible payoff matrix

- This time people are more incline to be altruistic

# of utiles, or utility:

(A,C) → 3 – 4 = -1
my 'A' - my guilt

(C, A) → -1 – 2 = -3
my 'C' - my indignation

This is a **coordination problem**

my pair

me		alpha	beta
	alpha	0, 0	-1, -3
	beta	-3, -1	1, 1

- What would you choose now?
  - No dominated strategy

→ ***Payoffs matter.*** (we will come back to this game later)

Another possible payoff matrix (2)

- Selfish vs. Altruistic
- What do you choose?

In this case, alpha still **dominates**

The fact I (selfish player) am playing against an altruistic player doesn't change my strategy, even by changing the other Player's payoff

Me (Selfish) \ my pair (Altruistic)	alpha	beta
alpha	0 , 0	3, -3
beta	-1,-1	1,1

Another possible payoff matrix (3)

- Altruistic vs. Selfish
- What do you choose?

•Do I have a dominating strategy?
•Does the other player have a dominating strategy?

By thinking of what my "opponent" will do I can decide what to do.

Me (Altruistic) rows; my pair (Selfish) columns:

	alpha	beta
alpha	0 , 0	-1, -1
beta	-3,3	1,1

→ ***Put yourself in other players' shoes and try to figure out what they will do***

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Game in normal form

	Notation	E.g.: grade game
Players	*i, j, …*	Me and my pair
Strategies	s_i: a particular strategy of player i s_{-i}: the strategy of everybody else except player i	alpha
	S_i: the set of possible strategies of player i	{alpha, beta}
	s: a particular play of the game “**strategy profile**” (vector, or list)	(alpha, alpha)
Payoffs	$u_i(s_1,\ldots, s_i,\ldots, s_N) = u_i(s)$	$u_i(s)$ = see payoff matrix

Assumptions

- We assume all the ingredients of the game to be known
  - Everybody knows the possible strategies everyone else could choose
  - Everybody knows everyone else's payoffs

- This is not very realistic, but things are complicated enough to give us material for this class

Strict dominance

Definition: Strict dominance

We say player i's strategy s_i' is **strictly** dominated by player i's strategy s_i if:

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \textbf{ for all } s_{-i}$$

No matter what other people do, by choosing s_i instead of s_i' , player i will always obtain a higher payoff.

Example 1

			2	
		L	C	R
1	T	**5, -1**	**11, 3**	**0,0**
	B	6, 4	0, 2	2, 0

Players	1, 2	
Strategy sets	S_1={T,B}	S_2={L,C,R}
Payoffs	U_1(T,C) = 11	U_2(T,C) = 3

NOTE: This game is not symmetric

Example 2: “Hannibal” game

- An invader is thinking about invading a country, and there are 2 ways through which he can lead his army.
- You are the defender of this country and you have to decide which of these ways you choose to defend: you can only defend one of these routes.
- One route is a hard pass: if the invader chooses this route he will lose one battalion of his army (over the mountains).
- If the invader meets your army, whatever route he chooses, he will lose a battalion

Example 2: “Hannibal” game

		attacker: e	attacker: h
defender	E	**1, 1**	**1, 1**
	H	0, 2	2, 0

e, E = easy ; h,H = hard

- Attacker’s payoffs is how many battalions he will arrive with in your country
  - Defender’s payoff is the complementary to 2
- You are the defender, what do you do?

Weak dominance

Definition: Weak dominance

We say player i's strategy s_i' is **weakly** dominated by player i's strategy s_i if:

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \textbf{ for all } s_{-i}$$
$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \textbf{ for some } s_{-i}$$

No matter what other people do, by choosing s_i instead of s_i' , player i will always obtain a payoff at least as high and sometimes higher.

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The “Pick a Number” Game

Without showing your neighbor what you’re doing, write down an integer number between 1 and 100. I will calculate the average number chosen in the class. The winner in this game is the person whose number is closest to two-thirds of the average in the class. The winner will win 5 euro minus the difference in cents between her choice and that two-thirds of the average.

Example: 3 students

Numbers: 25, 5, 60

Total: 90, Average: 30, 2/3*average: 20

25 wins: 5 euro – 5cents = 4.95 euro

First reasoning

- A possible assumption:
  - People chose numbers uniformly at random

➔The average is 50

➔2/3 * average = 33.3

- What's wrong with this reasoning?

Rationality: dominated strategies

- Are there dominated strategies?
- If everyone would chose 100, then the winning number would be 66

➔ numbers > 67 are weakly dominated by 66

➔ Rationality tells not to choose numbers > 67

Knowledge of rationality

- So now we've eliminated dominated strategies, it's like the game was to be played over the set [1, …, 67]
- Once you figured out that nobody is going to chose a number above 67, the conclusion is

➔Also strategies above 45 are ruled out

➔They are weakly dominated, only once we delete 68-100

- This implies rationality, and knowledge that others are rational as well

Common knowledge

- **Common knowledge**: you know that others know that others know … and so on that rationality is underlying all players’ choices

- … 1 was the winning strategy!!

- In practice:
  - Average was: Winning was: 2/3*average

- Now let’s play again!

Warning on iterative deletion

- **Iterative deletion of dominated strategies** seems a powerful idea, but it's also dangerous if you take it literally
- In some games, iterative deletion converges to a single choice, in others it may not (see Osborne-Rubinstein)

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A simple model in politics

- **2 candidates** choosing their political positions on a **spectrum**
- Assume the spectrum has 10 positions, with 10% voters on each
- Assume voters vote for closest candidate and break ties by splitting votes equally
- Candidate's payoff = share of votes

1 2 3 4 5 6 7 8 9 10

LEFT WING RIGHT WING

Dominated strategies

- Is position 1 dominated?
  - Testing domination by 2

Vs. 1	$u_1(1,1)$ = 50 %	<	$u_1(2,1)$ = 90%
Vs. 2	$u_1(1,2)$ = 10 %	<	$u_1(2,2)$ = 50%
Vs. 3	$u_1(1,3)$ = 15 %	<	$u_1(2,3)$ = 20%
Vs. 4	$u_1(1,4)$ = 20 %	<	$u_1(2,4)$ = 25%
…	…	…	….

- Same reasoning → 9 strictly dominates 10

Other dominated strategies?

- Is 2 dominated by 3?

- Can we go further?

The Median Voter Theorem

- Continuing the process of iterative deletion
  - Only positions 5 and 6 remain

➔Candidates will be squeezed towards the center, i.e., they will choose positions very close to each other

In political science this is called the

Median Voter Theorem

The Median Voter Theorem

- Other application in economics: **product placement**
- Example:
  - You are placing a gas station
  - you might think that it would be nice if gas stations spread themselves evenly out over the town, or on every road, so that there would be a station close by when you run out of gas
- As we all know, this doesn't happen: all gas stations tend to crowd into the same corners, all the fast foods crowd as well, etc.

Critics

- We used a model of a real-world situation, and tried to predict the outcome using game theory
- The model is simplified: it misses many features!
  - Voters are not evenly distributed
  - Many voters do not vote
  - There may be more than 2 candidates

- So is this model (and modeling in general) useless?
- No! First, analyze a problem with simplifying assumptions, then relax them and see what happens
  - E.g.: would a different voters distribution change the result?

- We will see throughout the course (and in the NetEcon course) examples of simplified model giving very useful predictions

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Example

		Player 2 l	c	r
	U	**0,4**	**4,0**	**5,3**
Player 1	M	4,0	0,4	5,3
	D	3,5	3,5	6,6

- Is there any dominat**ed** strategy for player 1/2?
- What would player 1 do if player 2 plays
  - left?
  - center?
  - right?
- What would player 2 do if player 1 plays
  - Up?
  - Middle?
  - Down?

Best response definition

Definition: Best Response

Player *i'*s strategy $\hat{s}_i$ is a BR to strategy s_{-i} of other players if:

$$u_i(\hat{s}_i , s_{-i}) \geq u_i(s'_i , s_{-i}) \text{ for all } s'_i \text{ in } S_i$$

or

$$\hat{s}_i \text{ solves } \mathbf{max}\ u_i(s_i , s_{-i})$$

Best responses in the simple game

		Player 2		
		l	c	r
Player 1	U	0,4	4,0	5,3
	M	4,0	0,4	5,3
	D	3,5	3,5	6,6

- $BR_1(l) = M$ $BR_2(U) = l$
- $BR_1(c) = U$ $BR_2(M) = c$
- $BR_1(r) = D$ $BR_2(D) = r$

- Does this suggest a solution concept?

Nash equilibrium definition

Definition: Nash Equilibrium

A strategy profile (s_1*, s_2*,…, s_N*) is a Nash Equilibrium (NE) if, for each i, her choice s_i* is a best response to the other players' choices s_{-i}*

- On of the most important concept in game theory
  - Used in many applications
- Seminal paper J. Nash (1951)
  - Nobel 1994

Nash equilibrium in the simple game

Player 2

		l	c	r
Player 1	U	0,4	4,0	5,3
	M	4,0	0,4	5,3
	D	3,5	3,5	6,6

- $BR_1(l) = M$ $BR_2(U) = l$
- $BR_1(c) = U$ $BR_2(M) = c$
- $BR_1(r) = D$ $BR_2(D) = r$

- (D, r) is a NE

NE motivation

- Real players don’t always play NE but
- No regret: Holding everyone else’s strategies fixed, no individual has a ***strict*** incentive to move away
  - Having played a game, suppose you played a NE: looking back the answer to the question “Do I regret my actions?” would be “No, given what other players did, I did my best”
  - Sometimes used as a definition: a NE is a profile such that no player can strictly improve by unilateral deviation
- Self-fulfilling belief:
  - If I believe everyone is going to play their parts of a NE, then everyone will in fact play a NE
- We will see other motivations

Remark: Best response may not be unique

		Player 2		
		l	c	r
	U	**0,2**	**2,3**	**4,3**
Player 1	M	11,1	3,2	0,0
	D	0,3	1,0	8,0

- Find all best responses
- Find NE

NE vs. strict dominance

		Player 2	
		alpha	beta
Player 1	alpha	**0,0**	**3,-1**
	beta	-1,3	1,1

- What is this game?
- Find NE and dominated strategies.

➔**No strictly dominated strategies could ever be played in NE**

– Indeed, a strictly dominated strategy is never a best response to anything

NE vs. weak dominance

- Can a weakly dominated strategy be played in NE?
- Example:

Player 2

Player 1

	l	r
U	**1,1**	**0,0**
D	0,0	0,0

- Are there any dominated strategies?
- Find NE
- Conclude

Summary of lecture 1

- Basic concepts seen in this lecture
  - Game in normal form
  - Dominated strategies (strict, weak), iterative deletion
  - Best response and Nash equilibrium
- Game theory is a mathematical tool to study strategic interactions, i.e., situations where an agent's outcome depends not only on his own action but also on other agents' actions
  - Many applications (we will see some)
  - Understand the world

Remark

- In most of the games seen in this lecture, the action sets were finite (i.e., players had a finite number of actions to choose from)
- This is not a general thing: we will see many games with continuous action sets (exercises and next lectures)
  - Example: companies choosing prices