Final exam

Patrick Loiseau

Game Theory, Fall 2015

2 hours, no document allowed except an A4 sheet of paper (both sides) with handwritten notes only.

Exercise 1 (~ 5 points)

Consider the symmetric game with the following payoffs (in which $a \leq 2$ is a parameter):

	U	D
U	a, a	$3, 0$
D	$0, 3$	$2, 2$

1. Assume that $a > 0$. Find all Nash equilibria and all evolutionary stable strategies.

   Answer: The only NE is (U, U) and it is strict so it is an ESS.

2. Assume that $a = 0$. Find all Nash equilibria in pure strategies and all pure evolutionary stable strategies.

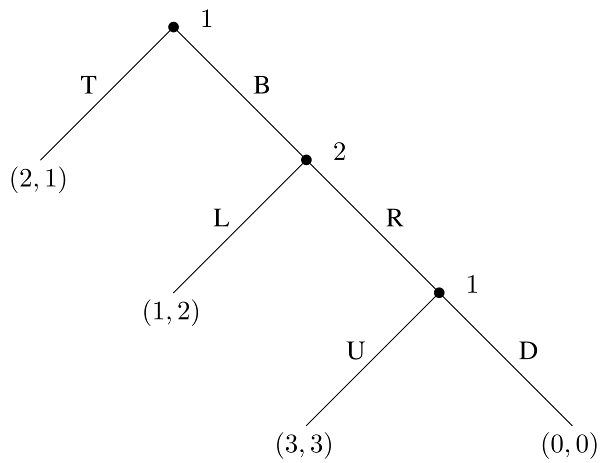
   Answer: (U, U), (U, D) and (D, U) are NE but not strict. U is still an ESS because $u(U, D) = 3 > u(D, D) = 2$**.**

3. Assume that $a < 0$. Find all Nash equilibria.

   Answer: (U, D), (D, U) and $(1/(1-a), -a/(1-a))$**.**

Exercise 2 (~ 6 points)

Consider the following game in extensive form. On the nodes where 1 (respectively 2) is written, player 1 (respectively 2) moves. For each outcome of the game, the first number represents the utility of player 1 and the second number the utility of player 2.



1. Apply backward induction.

   Answer: B, R, U

2. Write the game in strategic form.

   Answer:

	L	**R**
TU	$2, 1$	$2, 1$
TD	$2, 1$	$2, 1$
BU	$1, 2$	$3, 3$
BD	$1, 2$	$0, 0$

3. Find all pure Nash equilibria. Which ones are sub-game perfect?

   Answer: All pure NE: $\{$TU, L$\}$, $\{$TD, L$\}$, $\{$BU, R$\}$. Only the last is sub-game perfect.

4. Is there a pure Nash equilibrium which pareto dominates the other pure Nash equilibria?

   Answer: Yes, $\{$BU, R$\}$.

Exercise 3 (~ 9 points)

We consider the following public good provision game. There are 2 players, each choosing the amount of money x_i ($i \in \{1, 2\}$) they will give to build a public good. We assume that each player has a maximum of 1 unit of money that he can give, so that $x_i \in [0, 1]$ for both players. Once the good is built, they receive a utility $h(G)$ from using it, where

$G = x_1 + x_2$ is the total amount that was invested in the public good. We assume that $h(G) = KG^\alpha$, where $K \geq 0$ and $\alpha \in (0,1)$ are constants. Each players utility is therefore

$$u_i(x_1, x_2) = K(x_1 + x_2)^\alpha - x_i \quad (i \in \{1,2\}). \tag{1}$$

1. For a given value of $x_1 \in [0,1]$, compute the best response of player 2. Give also the best response of player 1 to $x_2 \in [0,1]$.

   Answer: We observe at first that the game is symmetric.

   Let $x_1 \in [0,1]$. The utility $u_2(x_1, x_2)$ as a function of x_2 is strictly concave. The FOC (first order condition) is given by

$$\frac{\partial u_2}{\partial x_2} = \alpha K (x_1 + x_2)^{\alpha - 1} - 1 = 0,$$

   i.e., $x_2 = (\alpha K)^{1/(1-\alpha)} - x_1$. Remembering that also x_2 must be in $[0,1]$, we include the border conditions and we obtain that the best response of player 2 to x_1 is

$$BR_2(x_1) = \begin{cases} 0 & \textbf{if} \quad x_1 > (\alpha K)^{1/(1-\alpha)} \\ 1 & \textbf{if} \quad x_1 < (\alpha K)^{1/(1-\alpha)} - 1 \\ (\alpha K)^{1/(1-\alpha)} - x_1 & \textbf{otherwise.} \end{cases}$$

   Symmetrically, the best response of player 1 to x_2 is

$$BR_1(x_2) = \begin{cases} 0 & \textbf{if} \quad x_2 > (\alpha K)^{1/(1-\alpha)} \\ 1 & \textbf{if} \quad x_2 < (\alpha K)^{1/(1-\alpha)} - 1 \\ (\alpha K)^{1/(1-\alpha)} - x_2 & \textbf{otherwise.} \end{cases}$$

2. Draw the best response diagram in the three cases $K \in [0, \frac{1}{\alpha}]$, $K \in [\frac{1}{\alpha}, \frac{1}{\alpha}2^{1-\alpha}]$ and $K \geq \frac{1}{\alpha}2^{1-\alpha}$.

   Answer: See Figure 1.

3. Give all Nash equilibria in pure strategy [hint: separate the cases $K \in [0, \frac{1}{\alpha}]$, $K \in [\frac{1}{\alpha}, \frac{1}{\alpha}2^{1-\alpha}]$ and $K \geq \frac{1}{\alpha}2^{1-\alpha}$].

   Answer: The NE are the points where the best responses intersect. From the graphs of the previous question we get:

Case 1 **If $K \in [0, \frac{1}{\alpha}]$, the NE are all the profiles of the form $(x_1, (\alpha K)^{1/(1-\alpha)} - x_1)$ with $x_1 \in [0, (\alpha K)^{1/(1-\alpha)}]$.**

Case 2 **If $K \in [\frac{1}{\alpha}, \frac{1}{\alpha}2^{1-\alpha}]$, the NE are all the profiles of the form $(x_1, (\alpha K)^{1/(1-\alpha)} - x_1)$ with $x_1 \in [(\alpha K)^{1/(1-\alpha)} - 1, 1]$.**

Case 3 **If $K \geq \frac{1}{\alpha}2^{1-\alpha}$, the only NE is $(1,1)$.**

4. Suppose that there is a social planner that can choose both x_1 and x_2 in order to maximize $u_1(x_1, x_2) + u_2(x_1, x_2)$. What values could he choose (give all possible solutions)? [hint: separate different regions depending on the value of K, but not the same regions as in the previous question.]

   Answer: We may write the aggregate utility as $u_1(x_1, x_2) + u_2(x_1, x_2) = 2K(x_1 + x_2)^\alpha - x_1 - x_2 =$

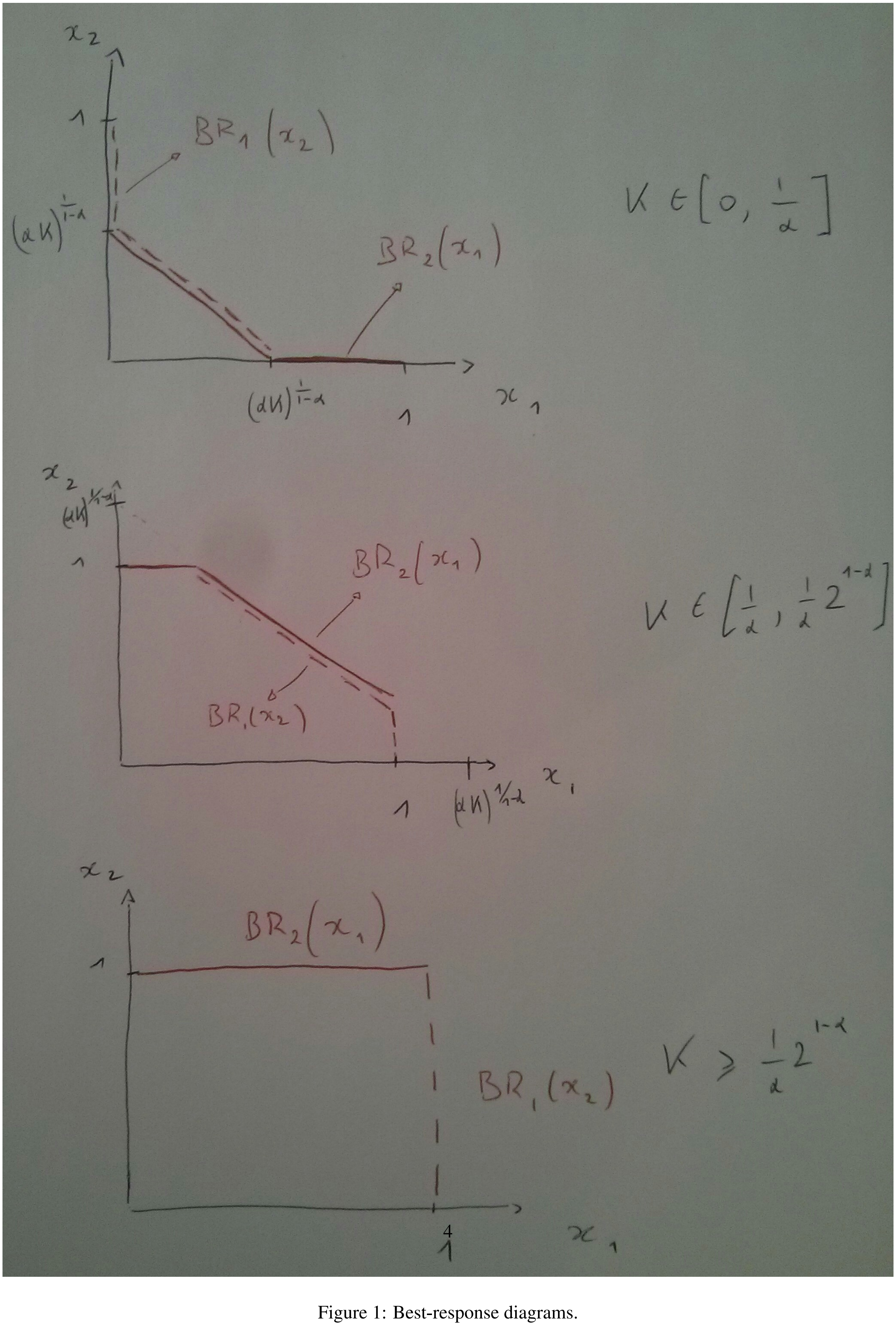


Figure 1: Best-response diagrams.

$U(x_1, x_2)$. This is a concave function. The FOC are given by

$$\frac{\partial U}{\partial x_1} = 2\alpha K(x_1 + x_2)^{\alpha-1} - 1 = 0$$
$$\frac{\partial U}{\partial x_2} = 2\alpha K(x_1 + x_2)^{\alpha-1} - 1 = 0,$$

then they have as solution any profile which satisfies $x_1 + x_2 = (2\alpha K)^{1/(1-\alpha)}$. Imposing the border conditions, we obtain that the social planner can maximize the aggregate utility by choosing:

Case 1 **If $K \in [0, \frac{1}{2\alpha}]$, any profile $(x_1, (2\alpha K)^{1/(1-\alpha)} - x_1)$ with $x_1 \in [0, (2\alpha K)^{1/(1-\alpha)}]$;**

Case 2 **If $K \in [\frac{1}{2\alpha}, \frac{2^{-\alpha}}{\alpha}]$, any profile $(x_1, (2\alpha K)^{1/(1-\alpha)} - x_1)$ with $x_1 \in [(2\alpha K)^{1/(1-\alpha)} - 1, (2\alpha K)^{1/(1-\alpha)}]$;**

Case 3 **If $K > \frac{2^{-\alpha}}{\alpha}$, only the profile $(1, 1)$.**

5. Compare the answer of question 4. to the Nash equilibria and comment.

Answer: The amount of public good achieved at Nash equilibrium is never larger, and sometimes strictly smaller (when $K < \frac{1}{\alpha}2^{1-\alpha}$) than at social optimum. This is due to externalities that are not taken into account for individual decision at Nash equilibrium.

6. Suppose now that $\alpha = 1$. Find all Nash equilibria in pure strategy.

Answer: The FOC condition is $K - 1 = 0$. If $K < 1$, $(0, 0)$ is the only NE. If $K > 1$, $(1, 1)$ is the only NE. If $K = 1$, any profile x_1, x_2 is a NE.