Final exam

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2 hours, no document allowed except an A4 sheet of paper (both sides) with handwritten notes only.

Questions (1 point each)

Say whether the following assertions are true or false and justify your answer in 1 short paragraph. In many cases, an example is sufficient.

1. A strictly dominated strategy can be played with positive probability in a Nash equilibrium strategy.

   Answer: False. If a strictly dominated strategy is played with positive probability, the player can improve his payoff by reducing the probability of playing the dominated strategy and increasing the probability of playing the dominating strategy.

2. A weakly dominated strategy can be played with positive probability in a Nash equilibrium strategy.

   Answer: True. For example in the following game:

	a	**b**
A	$1,1$	$0,0$
B	$0,0$	$0,0$

   B is dominated by A but (B,b) is a Nash equilibrium.

3. A mixed strategy where one player plays 2 actions with positive probability can be a strict Nash equilibrium.

   Answer: False. If a player plays 2 actions with positive probability at a Nash equilibrium, then these 2 pure actions give the same payoff and changing his strategy by modifying the probability of the two actions yields the same payoff.

4. A symmetric Nash equilibrium in a symmetric 2 players game cannot be an evolutionary stable strategy if it is a weak Nash equilibrium.

   Answer: False. See definition 2 of the ESS. For example, in the game

	A	**B**
A	$1,1$	$1,1$
B	$1,1$	$0,0$

A is an ESS and (A, A) is a weak NE.

Exercise A (~ 8 points)

We consider the following public good provision game. There are 2 players, each choosing the amount of money x_i ($i \in \{1, 2\}$) they will give to build a public good. We assume that each player has a maximum of 1 unit of money that he can give, so that $x_i \in [0, 1]$ for both players. Once the good is built, they receive a utility $h(G)$ from using it, where $G = x_1 + x_2$ is the total amount that was invested in the public good. We assume that $h(G) = K\sqrt{G}$, where $K \geq 0$ is a non-negative constant. Each players utility is therefore

$$u_i(x_1, x_2) = K\sqrt{x_1 + x_2} - x_i \quad (i \in \{1, 2\}). \tag{1}$$

1. For a given value of $x_1 \in [0, 1]$, compute the best response of player 2 (hint: be careful that it must be in $[0, 1]$). Give also the best response of player 1 to $x_2 \in [0, 1]$.

   Answer: The best response of player 2 to x_1 is

$$x_2^r = \begin{cases} 0 & \textbf{if} \quad x_1 > K^2/4 \\ 1 & \textbf{if} \quad x_1 < K^2/4 - 1 \\ K^2/4 - x_1 & \textbf{otherwise.} \end{cases}$$

   Symmetrically, the best response of player 1 to x_2 is

$$x_1^r = \begin{cases} 0 & \textbf{if} \quad x_2 > K^2/4 \\ 1 & \textbf{if} \quad x_2 < K^2/4 - 1 \\ K^2/4 - x_2 & \textbf{otherwise.} \end{cases}$$

2. Draw the best response diagram in the three cases $K \in [0, 2]$, $K \in [2, 2\sqrt{2}]$ and $K \geq 2\sqrt{2}$.

   Answer: See Figure ??.

3. Give all Nash equilibria in pure strategy (hint: separate the cases $K \in [0, 2]$, $K \in [2, 2\sqrt{2}]$ and $K \geq 2\sqrt{2}$).

   Answer: The NE are the points where the best response intersect. From the graphs of the previous question we get:

Case 1 If $K \in [0, 2]$, the NE are all the profiles of the form $(x_1, K^2/4 - x_1)$ with $x_1 \in [0, K^2/4]$.

Case 2 If $K \in [2, 2\sqrt{2}]$, the NE are all the profiles of the form $(x_1, K^2/4 - x_1)$ with $x_1 \in [K^2/4 - 1, 1]$.

Case 3 If $K > 2\sqrt{2}$, the only NE is $(1, 1)$.

4. Suppose that there is a social planner that can choose both x_1 and x_2 in order to maximize $u_1(x_1, x_2) + u_2(x_1, x_2)$. What values could he choose? (give all possible solutions)

   Answer: By differentiating, we find that, to maximize $u_1(x_1, x_2) + u_2(x_1, x_2)$, the social planner can choose any profile (x_1, x_2) such that $x_1 + x_2 = K^2$. More precisely,

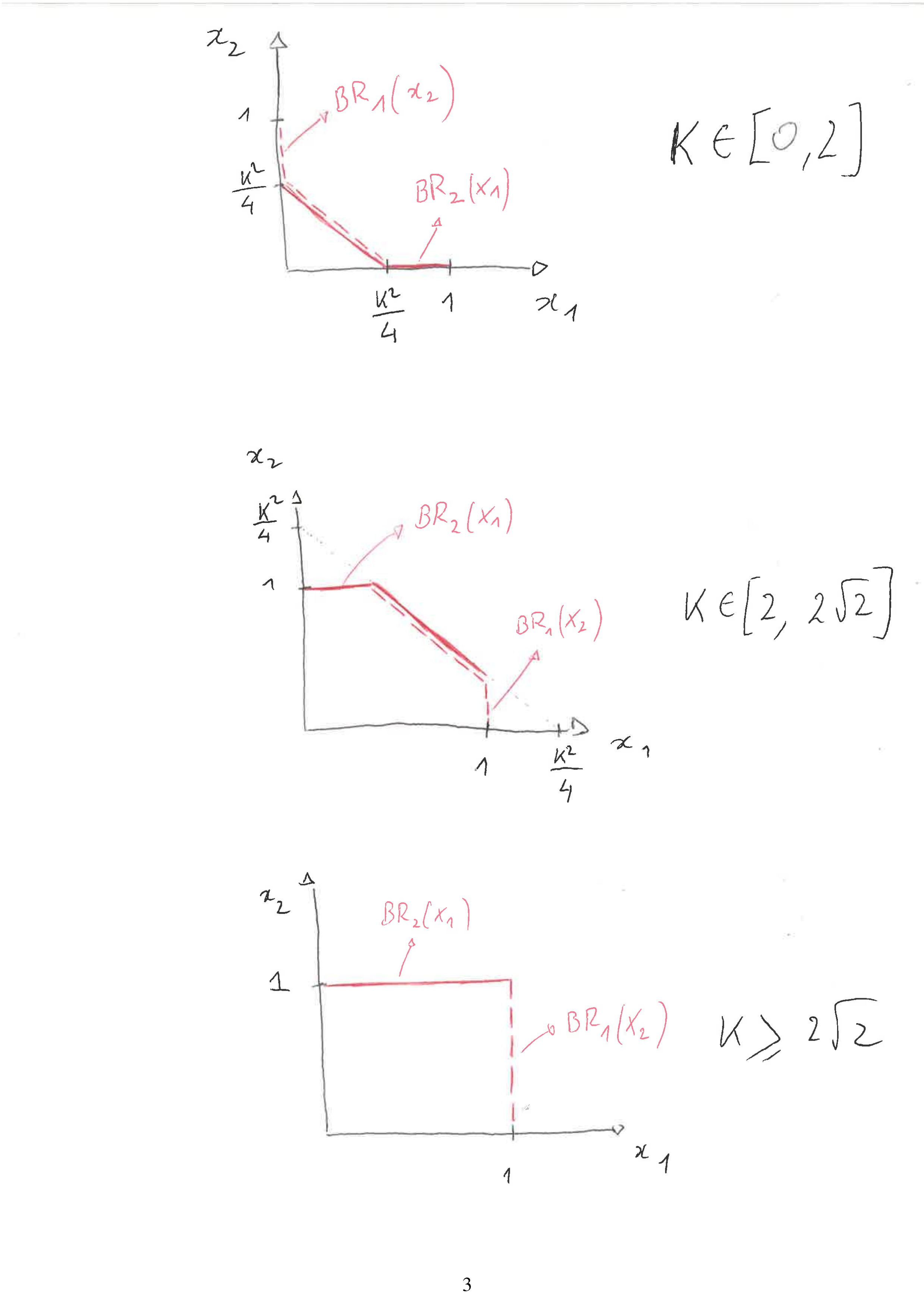

Figure 1: Best-response diagrams.

Case 1 **If $K \in [0, 1]$, the social planner can choose any profile $(x_1, K^2 - x_1)$ with $x_1 \in [0, K^2]$;**

Case 2 **If $K \in [1, \sqrt{2}]$, the social planner can choose any profile $(x_1, K^2 - x_1)$ with $x_1 \in [K^2 - 1, K^2]$;**

Case 3 **If $K > \sqrt{2}$, the social planner can choose only the profile $(1, 1)$.**

5. Compare the answer of question 4. to the Nash equilibria and comment.

   Answer: The amount of public good achieved at Nash equilibrium is never larger, and sometimes strictly smaller (when $K < 2\sqrt{2}$) than at social optimum. This is due to externalities that are not taken into account for individual decision at Nash equilibrium.

Exercise B (~ 8 points)

Consider the following two-player game:

	l	r
T	α, α	$0, \gamma$
B	$\gamma, 0$	β, β

where $\alpha > \beta > 0$ are fixed parameters and $\gamma > 0$ is a parameter which can take on different (positive) values. In this exercise we study the equilibria for different values of γ.

1. First assume that $\gamma > \alpha$.

   (a) Is there a strictly dominated strategy?

   Answer: l is strictly dominated by r and T is strictly dominated by B.

   (b) Find all Nash equilibria (pure and mixed).

   Answer: The only NE is (B, r) (pure). Remark: we have seen that a NE in strictly dominating strategy is necessarily unique; there is no point in looking for a mixed-strategy NE (which you will of course not find).

   (c) What type of game is it?

   Answer: It is a prisoner's dilemma.

2. Assume now that $\gamma \leq \alpha$.

   (a) Is there a strictly dominated strategy? Is there a weakly dominated strategy?

   Answer: There is no strictly dominated strategy. If $\gamma = \alpha$, l is weakly dominated by r and T is weakly dominated by B. If $\gamma < \alpha$, there is no weakly dominated strategy.

(b) Find all Nash equilibria (pure and mixed).

Answer: There are two pure strategy equilibria (T, l) and (B, r) and one mixed-strategy NE with the mix $(\beta/(\alpha - \gamma + \beta), (\alpha - \gamma)/(\alpha - \gamma + \beta))$ for each player. If $\gamma = \alpha$, the mixed NE is the same as (T, l).

(c) For each Nash equilibrium (pure and mixed), say if it is strict or weak and under which condition on γ.

Answer: The mixed-strategy NE is always weak. The NE (B, r) is always strict. The NE (T, l) is strict if $\gamma < \alpha$ and weak if $\gamma = \alpha$.

(d) Among the pure strategy equilbria, say if one Pareto dominates the other.

Answer: (T, l) Pareto dominates (B, r).

(e) Say which strategies (pure only) are evolutionary stable and under which condition on γ.

Answer: Remark that the game is symmetric. To discuss ESS, we interpret the actions as $B = r$ and $T = l$. (B, r) is always a strict NE so B is an ESS. Similarly, T is an ESS if $\gamma < \alpha$ since (T, l) is a strict NE. If $\gamma = \alpha$, (T, l) is a weak NE. We have $u_1(T, r) = 0 < u_1(B, r) = \beta$, so T is not an ESS.

(f) Suppose that $\gamma < \alpha$. Is the mixed strategy played at Nash equilibrium an evolutionary stable strategy?

Answer: Let $\hat{s} = (\frac{\beta}{\alpha-\gamma+\beta}, \frac{\alpha-\gamma}{\alpha-\gamma+\beta})$ be the mixed strategy NE. It is a weak NE. For $\hat{s}$ to be a NE, we must have $u_1(\hat{s}, s') > u_1(s', s')$ for all $s' \neq \hat{s}$. But for $s' = r$ (pure strategy), $u_1(\hat{s}, s') = \frac{\alpha-\gamma}{\alpha-\gamma+\beta} \cdot \beta < u_1(s', s') = \beta$. Therefore $\hat{s}$ is not an ESS.